PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Effect of a Sudden Change of Shape of the Bottom of a Slightly Compressible Ocean

C. C. L. Sells

Phil. Trans. R. Soc. Lond. A 1965 **258**, 495-528
doi: 10.1098/rsta.1965.0049

THE EFFECT OF A SUDDEN CHANGE OF SHAPE OF THE BOTTOM OF A SLIGHTLY COMPRESSIBLE OCEAN

By C. C. L. SELLS
Sidney Sussex College, Cambridge

(*Communicated by G. K. Batchelor, F.R.S.—Received* 5 *August* 1964)

CONTENTS

When a block on the rigid bottom of a layer of slightly compressible gravitating liquid is instantaneously jerked upwards, acoustic and gravity waves are developed. In this paper a two-dimensional problem is set up, with an infinite-strip block in a layer of uniform depth. There are two distinct regions in space and time; in one (the first in time), the acoustic disturbance predominates with gravity entering as a perturbation, in the other the gravity disturbance is dominant, with compressibility producing a small correction to the motion. In both cases, the ratio of the velocities of long gravity waves and of sound in the medium characterizes the perturbation terms.

The form of the acoustic pulses from a source of this type appears to be of theoretical interest, as point and line sources are usually treated in the literature. The gravity waves are akin to Cauchy–Poisson waves and have been much studied in connexion with tsunamis and ocean waves in general, but the treatment of the compressibility effect is thought to be original.

Introduction

The aim of the present work is to study the disturbance at the free surface of a gravitating and slightly compressible liquid layer (ocean) of constant finite depth, when a block of the solid bottom is suddenly jerked upwards through a small distance. The problem is treated as two-dimensional, that is, the block is taken to be a very long strip of constant width.

This form of initial disturbance is a possible representation of the sea-bed earthquake or volcanic action which generates tsunamis, or long gravity waves of destructive effect; indeed, it is essentially the form used by Hendrickson (1962) in his appendix to a general survey of tsunamis (Wilson, Webb & Hendrickson 1962). Stoneley (1963) following Jeffreys & Jeffreys (1956, §17.09), uses a similar block elevation at the water surface, and derives a result for the surface displacement at the head of the gravity wave train, modified slightly by compressibility. Takahasi (1947) derives the formal solution (without compressibility) for a general two-dimensional bed disturbance and then assumes that the width of the disturbance is very large (of order 100 km) compared with the depth of the ocean and accordingly evaluates the integrals very approximately. Hendrickson (1962) works on the opposite hypothesis, namely that the width is small compared with the depth.

Earthquake fault lengths of the order of tens of kilometres, or many times the ocean depth, are known, so this representation of an ocean bed disturbance as a long strip is not necessarily unrealistic, although the solution, regarded as a function of the disturbance, will be of demonstrable value only at a distance from the strip small compared with its length. This need not be inconsistent with the requirement, posed in later sections of this paper, that this distance be large compared with the layer depth. Moreover, we later derive a result for the compressible-fluid effect on the velocity of the gravity wavefront which depends only on the layer depth and not on any other length scale of the problem, and we may expect that the strip length will likewise have little effect on this velocity. However, in the realm of finite initial disturbances, Unoki & Nakano (1953) have considered the Cauchy–Poisson problem for circular disturbances at the surface (similar to Stoneley in this respect), of both impulse and elevation type, and they examine two distributions of impulse over the surface, block and smoothly heaped; they ignore compressibility.

It is almost certain that the locality of the recording station is of great importance in determining the essential characteristics of a distant tsunami record. Unoki & Nakano (1953) successfully isolated some features of their records and showed that the initial disturbance (assumed to be at the surface) was of impulse rather than elevation type; moreover, they were able to distinguish between block or heaped-up distributions by the presence or absence of beats in the record; but their tsunamis were generated comparatively locally, and presumably the effect of the run-up to the station was not as marked as it would have been at a more distant station. Munk (1962), discussing distant tsunami readings, has this to say: 'In general it is found that the spectra of different tsunamis at any one station look alike, whereas one tsunami at different stations has no reproducible spectral features. The inevitable conclusion is that tsunami records are governed principally by the bottom topography near the recording station, and not by the character of the source.'

Thus the determination of the type of earthquake responsible for a given tsunami is difficult except from local or sea station records.

Little seems to have been written on compression waves in the ocean. Pidduck (1912) gave the formal solution, as an integral, for a point source in a liquid layer of finite depth, and showed that the Cauchy–Poisson formula is obtained by passing to the limit of infinite sound velocity, so that the disturbance manifests itself instantaneously at any point. Bondi (1947) investigated the effects of gravity and surface tension on a compressible fluid half-space, with a point source of waves at a finite depth; he found that 'faster-than-sound' propagation along the bounding free surface is possible under the action of surface tension, but that no such phenomenon can occur under gravity action alone. As remarked above, Stoneley (1963) has noted the first-order effect of the sound velocity on the gravity wave near the front. In Bondi's work the shape of the compression wave (and its reflexion in the boundary surface) is given by elementary acoustic theory, but the other authors do not treat the direct and reflected waves which are the *sine qua non* of seismology, beside the various surface and interface waves which may arise. Stoneley (1963) comments that the sound waves will presumably show up as the T-phase on a seismogram, but does not investigate them further. However, the compressible ocean is taken into account in the theory of sub-oceanic Rayleigh waves (Stoneley 1926), which thus belong to the family of interface waves just mentioned.

In this work, it will be shown that the acoustic surface disturbance separates into two parts. There is in general a system of sound waves, direct and due to reflexion alternately at the free surface and the bottom, from the edges of the block disturbance (hereafter referred to as 'the block', for convenience), but for an observer directly over the block, another system arises from the point of it nearest to the observer. The first system is described in §2 (exact pure acoustic disturbance, with gravity correction) and in §3 (asymptotic expansion at very large distances), the second system is treated in §4. This second system is an alternating set of discontinuous jumps, again direct and reflected. Unoki & Nakano (1953) report that a vessel which happened to be passing over a submarine volcano at the moment of eruption received a violent 'sea shock'; this agrees with the predicted theoretical result for the direct wave. In practice the subsequent reflexions would not be as strong as in the theory, because of transmission of some energy as seismic waves and possibly owing to absorption at a soft layer on the sea-bed—both these factors reduce the reflexion coefficient at the bottom. Also in practice the initial shock is less severe than that given here, as the block is not instantaneously disturbed.

In the theory of the sound waves, we regard gravity as a small correction or perturbation near the beginning of each wave; the correction is calculated to the first-order in the parameter (gh/c^2) and its product with dimensionless time (ct/h), where h is the layer depth and c is the sound velocity, and gravity is otherwise regarded purely as a means of maintaining contact between the liquid and the bottom. We can describe the sound wave system as a pure acoustic disturbance, modified by a gravity perturbation. Since this perturbation involves the time, it increases in importance as time goes on, and eventually the acoustic disturbance is swamped by the gravity terms. For a heavy and slightly compressible liquid this occurs in a comparatively short time, so that the gravity waves will predominate rapidly and the acoustic disturbance will be hardly noticeable to an observer. Apart from this, we have the dissipative effects mentioned above for an acoustic reflexion at the bottom in a practical case such as the sea, which induce something of the order of exponential damping in the acoustic waves.

In both the acoustic wave and the following gravity wave treatment, the problem is fully linearized; this is satisfactory for not too large times, since the disturbance amplitude is assumed small compared with the depth. Thus, possible non-linear effects, such as a steepening, unpredicted by the linear theory, of the wave fronts are ignored.

After the sound waves come the gravity waves, modified only slightly by compressibility. In the incompressible case, we find that the surface initially heaps up over the block, but the displacement decays to zero at large distances as we would expect; then the heaped-up elevation spreads out into progressive waves in each direction; the development of the first wave is described by three terms in an expansion in power series of an appropriate dimensionless time variable (§6). Webb (1962) claims, without appending calculations, that in an incompressible fluid, on passing to the limit of an instantaneous block upthrust, the surface profile matches the block immediately; this idea seems to have been due to a consideration solely of the second system of 'sea shocks' mentioned above, and is an erroneous conclusion as the present calculation reveals. In fact, this paper shows that the result of adding all the compression arrivals together in both systems and proceeding to the limit of infinite sound velocity and zero time—this being the time at which all the arrivals come in together in this limit—is exactly the same as the formula for the initial elevation over the block in the incompressible theory.

At large distances, the gravity wave takes on more and more the appearance of an Airy function, exponentially small in front and oscillatory behind (§ 7). The leading wave propagates with the velocity $(gh)^{\frac{1}{2}}$ of long waves in liquid of depth h, with a decaying wave train behind it. The effect of compressibility is now of the first-order of small quantities, again involving the parameter (gh/c^2), and can be approximately inserted into the expression for the leading wave to show that the velocity of this wave is decreased by a factor $(1-\frac{1}{4}gh/c^2)$. Lastly, in § 8, the tail or coda of the wave is examined for times long after the passing of the leading wave, and is found to be an exponentially damped non-dispersive train of waves moving a little faster than ordinary long waves and with wavelength about 5·6 times the depth. The compression effect is again calculated, and is found to be an increase in the coda velocity, by a factor $\{1+0{\cdot}09\,gh/c^2\}$, together with a decrease in the damping coefficient with regard to time. The coda is not the limit for large time of the Airy function describing the leading wave; a transition takes place from one form to the other as the time increases beyond a certain value, depending on the observation point, but certainly greater than the leading wave travel time.

It will be noticed that in all steps of the solution—for both the acoustic disturbance modified by gravity, and the gravity wave modified by compressibility—expansion in powers of a dimensionless parameter (gh/c^2) is involved, and squares and higher powers have been neglected. The occurrence of g in the numerator makes the parameter relevant for the first type of disturbance—if g tends to zero, a pure acoustic problem develops (which has been completely solved in this paper); the factor c^2 in the denominator causes the parameter likewise to tend to zero as the sound velocity increases and becomes infinite in the limit of incompressible fluid with gravity waves only. The parameter is thus very useful in both types of approximate treatment of the problem.

1. The governing equations and the formal solution

We take a system of Cartesian co-ordinates (x, z) with the origin at the midpoint of the block on the ocean floor, the x axis in a horizontal direction perpendicular to the edge of the block, and the z axis vertically upwards (see figure 1). The total width of the block is $2a$, so that its edges are at $(\pm a, 0)$, the ocean will be taken as having uniform length h, and we suppose that the block is jerked suddenly upwards through unit distance at time $t=0$, so that the displacement of the sea-bed in the z-direction is given by

$$\begin{matrix} H(t) & |x|\leqslant a. \\ 0 & \text{otherwise} \end{matrix} \tag{1·1}$$

The linearized equation for the displacement potential ϕ, defining displacements $(\partial\phi/\partial x,\ \partial\phi/\partial z)$, in a gravitating compressible fluid with constant sound velocity c throughout its depth, is (Lamb 1957; Bondi 1947; Stoneley 1926)

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=\nabla^2\phi-2\gamma\frac{\partial\phi}{\partial z}, \tag{1·2}$$

where $$\gamma = g/2c^2. \tag{1·3}$$

We may put $$\phi = \psi\, e^{\gamma z} \tag{1·4}$$

and then the equation for ψ is $$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi - \gamma^2\psi. \tag{1·5}$$

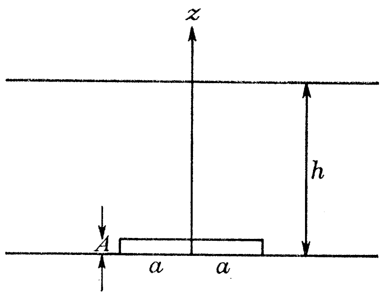

FIGURE 1. The physical situation.

Let us take an even Fourier transform in x and a Laplace transform in t,

$$\Psi = \int_0^\infty dx \cos(kx) \int_0^\infty dt\, e^{-pt}\,\psi. \tag{1·6}$$

Then from (1·5), Ψ satisfies the ordinary differential equation

$$\frac{d^2\Psi}{dz^2} = \left(k^2+\gamma^2+\frac{p^2}{c^2}\right)\Psi, \tag{1·7}$$

with the general solution $$\Psi = B(k,p)\, e^{\mu z} + C(k,p)\, e^{-\mu z}, \tag{1·8}$$

where $$\mu = (k^2+\gamma^2+p^2/c^2)^{\frac{1}{2}} \quad (\mathscr{R}\mu \geqslant 0). \tag{1·9}$$

B and C must now be chosen to satisfy the boundary conditions at the bottom $z = 0$ and at the free surface $z = h$. At the bottom, the vertical displacement

$$\partial\phi/\partial z = (\partial\psi/\partial z + \gamma\psi)\, e^{-\gamma z}$$

is given by the formulae (1·1); we take the double transform of these expressions, as given by (1·6), and note that the Laplace transform of $H(t)$ is $1/p$ and the even Fourier transform of the function $(0, |x| > a;\ 1, |x| < a)$ is

$$\int_0^a \cos(kx)\, dx = \frac{1}{k}\sin(ka). \tag{1·10}$$

Substituting (1·8) into the transformed boundary condition and putting $z = 0$, we find

$$(\mu+\gamma)\, B - (\mu-\gamma)\, C = \frac{1}{pk}\sin(ka). \tag{1·11}$$

At the free surface $z = h$, if the surface displacement is η, we have the kinematic condition

$$\eta = (\partial\phi/\partial z)_{z=h}, \tag{1·12}$$

and from Bernoulli's equation $$g\eta + (\partial^2\phi/\partial t^2)_{z=h} = 0, \tag{1·13}$$

so that $$g\frac{\partial\phi}{\partial z} + \frac{\partial^2\phi}{\partial t^2} = \left[g\left(\frac{\partial\psi}{\partial z}+\gamma\psi\right) + \frac{\partial^2\psi}{\partial t^2}\right] e^{\gamma z} = 0 \tag{1·14}$$

at $z = h$. The transform of this equation (1·14) with (1·8) yields

$$[p^2+g(\mu+\gamma)]\,e^{\mu h}B+[p^2-g(\mu-\gamma)]\,e^{-\mu h}C = 0. \qquad (1\cdot15)$$

The solution of equations (1·11), (1·15) is

$$B = \frac{[p^2-g(\mu-\gamma)]\,e^{-\mu h}}{(\mu-\gamma)\,[p^2+g(\mu+\gamma)]\,e^{\mu h}+(\mu+\gamma)\,[p^2-g(\mu-\gamma)]\,e^{-\mu h}}\,\frac{\sin(ka)}{pk} \qquad (1\cdot16)$$

$$C = -\frac{[p^2+g(\mu+\gamma)]\,e^{\mu h}}{(\mu-\gamma)\,[p^2+g(\mu+\gamma)]\,e^{\mu h}+(\mu+\gamma)\,[p^2-g(\mu-\gamma)]\,e^{-\mu h}}\,\frac{\sin(ka)}{pk}. \qquad (1\cdot17)$$

Now the inverse transform of (1·6) is

$$\psi = \frac{1}{2\pi i}\int_L dp\,e^{pt}\frac{2}{\pi}\int_0^\infty \Psi\,dk\cos(kx), \qquad (1\cdot18)$$

where L is the Bromwich contour in the complex p plane running from $\kappa-i\infty$ to $\kappa+i\infty$ in such a way that all the singularities of the integrand lie on the left. Then the surface displacement η is given from (1·4), (1·8), (1·13), (1·16) and (1·17) as

$$\eta = e^{\gamma h}\frac{1}{\pi^2 i}\int_L p\,dp\,e^{pt}\int_0^\infty \frac{dk}{k}\,\frac{\cos(kx)\sin(ka)}{p^2\cosh(\mu h)+\mu g\sinh(\mu h)-\gamma(\gamma g+p^2)\sinh(\mu h)/\mu}. \qquad (1\cdot19)$$

The remainder of this work is devoted to a discussion of η as given by (1·19). We shall find it very useful to consider the sine and cosine product as two sines:

$$\begin{aligned}\cos(kx)\sin(ka) &= \tfrac{1}{2}[\sin k(x+a)-\sin k(x-a)]\\ &= \tfrac{1}{2}[\sin k(a+x)+\sin k(a-x)]. \qquad (1\cdot20)\end{aligned}$$

We accordingly consider

$$\eta_X = e^{\gamma h}\frac{1}{2\pi^2 i}\int_L p\,dp\,e^{pt}\int_0^\infty \frac{dk}{k}\,\frac{\sin(kX)}{p^2\cosh(\mu h)+\mu g\sinh(\mu h)-\gamma(\gamma g+p^2)\sinh(\mu h)/\mu} \qquad (1\cdot21)$$

with the understanding that $X \geqslant 0$ and that when this function has been evaluated, η is to be found by adding the result for $X = a-x$ to that for $X = a+x$, when both these are positive, that is, when $|x| \leqslant a$ and the displacement is being observed at a point directly over the block; while, when $x > a$, η is to be found by subtracting the result for $X = x-a$ from that for $X = x+a$. This corresponds to all other points on the surface. The case $x < -a$ follows from $x > a$ since the disturbance is an even function of x.

2. The pure acoustic arrivals from the edges, with gravity perturbations

We consider the integral (1·21) for the surface displacement η_X.

From theoretical experience, we know that at least part of the integral will contain terms representing a disturbance travelling with the speed of sound in the liquid. The ocean is treated as an acoustic medium with gravity acting as the mechanism holding it in contact with the bottom and exerting only a perturbing influence on the acoustic waves. A parameter characterizing the perturbation is clearly γ times some characteristic length scale; this may be taken as h near $X = 0$, and may possibly be X when X/h is large, that is, at a great distance from the block. (Later on, observing the motion at a fixed X, we obtain the well known gravity waves in an almost incompressible medium. For long waves, their velocity is $(gh)^{\frac{1}{2}}$ and since the ratio of this velocity to c is small for most oceans, the two kinds of motion are well separated in time.)

It is accordingly convenient to expand the expression for η_X in powers of γ times some unknown length, or velocity-time, scale, and retain the zero-order term as the pure acoustic disturbance while the first-order term will be a measure of the gravity perturbation. We treat g similarly, since $g = 2c^2\gamma$ (but keep g and γ separate for convenience, for the moment). It is fortunate that this attack is available, since the factor $\mu = (\gamma^2+k^2+p^2/c^2)^{\frac{1}{2}}$ in the hyperbolic arguments causes great mathematical difficulties in exact treatment of the problem. The neglect of γ^2, however, renders it comparatively simple.

The method which we will apply in this section is to expand the integrand in (1·21) in powers of $e^{-\mu h}$ (wave expansion), as well as of γ, and then use Cagniard's (1939) technique to transform the double integrals into single convolution integrals the first few of which can be evaluated exactly. The idea is to write the double integrals essentially in the form of a Laplace transform and its inverse; Cagniard's original paper was applicable to a more complicated type of integral with two radicals, but the present case—with the simplifications described above—is straightforward.

We start by writing, in (1·21)

$$\sin(kX) = -\mathscr{I}\, e^{-ikX}. \tag{2·1}$$

Then the wave expansion gives

$$[p^2\cosh(\mu h)+\{\mu^2 g-\gamma(p^2+\gamma g)\}\sinh(\mu h)/\mu]^{-1}$$
$$= \frac{2}{p^2}\sum_{n=0}^{\infty}(-)^n\left\{1+(2n+1)\,\frac{(\gamma-g\mu^2/p^2)}{\mu}+O\left(\frac{\gamma^2}{\mu^2},\frac{\gamma^2c^2}{p^2},\frac{g^2\mu^2}{p^4}\right)\right\}e^{-(2n+1)\mu h}. \tag{2·2}$$

We now change the variable k to θ, to factor p out of the exponents, by

$$k = p\theta, \tag{2·3}$$

so that

$$\mu = (k^2+p^2/c^2)^{\frac{1}{2}} = p\nu, \tag{2·4}$$

and the nth exponent is

$$-ikX-(2n+1)\,\mu h = -p\{i\theta X+(2n+1)\,\nu h\}, \tag{2·5}$$

where

$$\nu = (\theta^2+1/c^2)^{\frac{1}{2}}. \tag{2·6}$$

Since (2·1) has introduced a simple pole singularity at the origin $k=0$ (now $\theta=0$) in the integrand (1·21), we must replace the lower limit 0 by a small real positive number δ in the θ-plane, take the imaginary part required by (2·1) and then the limit as $\delta\to 0$. The result of all these operations on (1·21) is

$$\eta_X = -e^{\gamma h}\frac{1}{2\pi^2 i}\int_L p\,dp\,e^{pt}.\frac{2}{p^2}\lim_{\delta\to 0}\mathscr{I}\int_\delta^\infty\frac{d\theta}{\theta}\sum_{n=0}^{\infty}(-)^n\exp -p[(2n+1)\,\nu h+i\theta X]$$
$$\times\{1+(2n+1)\,(\gamma-g\nu^2)/p\nu\}. \tag{2·7}$$

For brevity, write

$$H_n = (2n+1)\,h \tag{2·8}$$

and

$$R_n^2 = X^2+H_n^2. \tag{2·9}$$

We change the variable again, to ζ defined for each n by

$$\zeta = H_n\nu+iX\theta. \tag{2·10}$$

Solving for θ, we find:
$$R_n^2\theta = -iX\zeta + H_n(\zeta^2 - R_n^2/c^2)^{\frac{1}{2}}, \tag{2·11}$$
where the positive radical is to be taken for ζ real and $\zeta > R_n/c$;
$$R_n^2\frac{d\theta}{d\zeta} = \frac{H_n\zeta}{(\zeta^2 - R_n^2/c^2)^{\frac{1}{2}}} - iX, \tag{2·12}$$
$$\theta = 0 \quad \text{corresponds to} \quad \zeta = H_n/c, \tag{2·13}$$
$$R_n^2\nu = H_n\zeta - iX(\zeta^2 - R_n^2/c^2)^{\frac{1}{2}}. \tag{2·14}$$
The contour of integration Γ in the ζ plane is shown in figure 2.

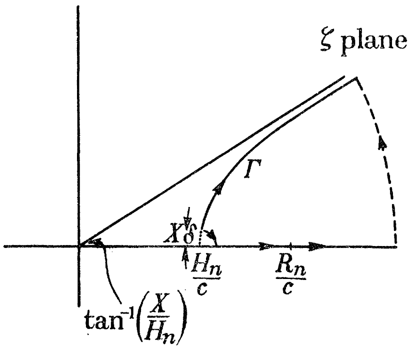

FIGURE 2. The Cagniard integration contour.

At $\zeta = H_n/c$, $d\zeta/d\theta = iX$, so the contour starts out in the direction of the imaginary axis. The point $\theta = \delta$ goes over into $\zeta = H_n/c + iX\delta$, to first-order in δ. As θ increases without limit, arg ζ approaches the value $\tan^{-1}(X/H_n)$, by (2·10), so Γ has an asymptote along the line shown.

We now distort Γ into the following path: a small quadrantal arc of radius $X\delta$ described in the negative direction, bringing us to the real axis; the real axis from this point
$$\zeta = H_n/c + X\delta \quad \text{to} \quad \zeta = +\infty;$$
and an arc at infinity connecting the real axis to Γ and the asymptote. The contribution to the integral from the arc at infinity vanishes because of the exponential factor $\exp(-p\zeta)$; the contribution from the small quadrantal arc is the residue at $\zeta = H_n/c$ multiplied by $(-\frac{1}{2}i\pi)$, and is denoted by η_B and evaluated in §4; the contribution from the real axis we shall call η_A.

Substituting (2·10) to (2·14) into (2·7), and making use of the fact that over the range $\zeta = H_n/c + X\delta$ to $\zeta = R_n/c$, the radical $(\zeta^2 - R_n^2/c^2)^{\frac{1}{2}}$ is pure imaginary for sufficiently small δ and $X \neq 0$, we find that the integrand is real over this segment of the real axis, so that there is no contribution to (2·7) from the segment and we may replace the lower limit of the integration by $\zeta = R_n/c$. Since this point is a branch point singularity only, we can integrate over it. The imaginary part can be extracted and simplified, and in the perturbation terms we use the relation $g = 2c^2\gamma$ to cancel two terms, but still retain γ and g separately; then the contribution from the real axis to η_X becomes
$$\eta_A = -e^{\gamma h}\frac{2}{\pi}\frac{1}{2\pi i}\int_L \frac{dp}{p}e^{pt}\sum_{n=0}^{\infty}(-)^n\left\{\frac{H_nX}{c^2}\int_{R_n/c}^{\infty}\frac{e^{-p\zeta}\,d\zeta}{(\zeta^2 - R_n^2/c^2)^{\frac{1}{2}}(\zeta^2 - H_n^2/c^2)}\right.$$
$$\left. + \frac{(2n+1)X}{p}\int_{R_n/c}^{\infty}\frac{[g(\zeta^2 - R_n^2/c^2)/R_n^2 - \gamma]\,\zeta e^{-p\zeta}\,d\zeta}{(\zeta^2 - R_n^2/c^2)^{\frac{1}{2}}(\zeta^2 - H_n^2/c^2)}\right\}. \tag{2·15}$$

Now, either by use of the convolution theorem for Laplace transforms or by interchanging the order of integration and carrying out the p integration, we have for $m \geqslant 0$

$$\frac{1}{2\pi i}\int_L \frac{e^{pt}\,dp}{p^{m+1}}\int_T^\infty f(\zeta)\,e^{-p\zeta}\,d\zeta = \frac{1}{m!}H(t-T)\int_T^t (t-\zeta)^m f(\zeta)\,d\zeta. \tag{2·16}$$

Hence

$$\eta_A = -\frac{2}{\pi}e^{\gamma h}\sum_{n=0}^{\infty}(-)^n H(t-R_n/c)\left\{\frac{H_n X}{c^2}\int_{R_n/c}^t \frac{d\zeta}{(\zeta^2-R_n^2/c^2)^{\frac{1}{2}}(\zeta^2-H_n^2/c^2)} + (2n+1)X\int_{R_n/c}^t \frac{[g(\zeta^2-H_n^2/c^2)/R_n^2-\gamma]\,\zeta(t-\zeta)\,d\zeta}{(\zeta^2-R_n^2/c^2)^{\frac{1}{2}}(\zeta^2-H_n^2/c^2)}\right\}$$

$$= -\frac{1}{\pi}e^{\gamma h}\sum_{n=0}^{\infty}(-)^n H\left(t-\frac{R_n}{c}\right)\left\{\tan^{-1}\frac{2H_n Xct\,(c^2t^2-R_n^2)^{\frac{1}{2}}}{c^2t^2(X^2-H_n^2)+R_n^2H_n^2} + (2n+1)\left[\frac{gX}{R_n^2}t\left(t^2-\frac{R_n^2}{c^2}\right)^{\frac{1}{2}} - \frac{gt}{c}\tan^{-1}\frac{(c^2t^2-R_n^2)^{\frac{1}{2}}}{X} + \frac{gH_n}{2c^2}\tan^{-1}\frac{2H_nXct(c^2t^2-R_n^2)^{\frac{1}{2}}}{c^2t^2(X^2-H_n^2)+R_n^2H_n^2}\right]\right\}. \tag{2·17}$$

The arctangents are to be taken between 0 and π.

The leading term in (2·17) is exactly the pure acoustic disturbance from the edge corresponding to X. We here recall that for $|x| < a$, the values for $X = a \pm x$ are added together, and for $x > a$, the value for $X = x-a$ is subtracted from that for $X = x+a$.

The behaviour of the pure acoustic part, which we may denote by $\eta_c e^{\gamma h}$ to remove the factor $e^{\gamma h}$, near the arrival times $t = R_n/c$ is given by

$$\eta_c \doteqdot -\frac{2^{\frac{3}{2}}}{\pi}\sum_{n=0}^{\infty}(-)^n H\left(t-\frac{R_n}{c}\right)\frac{c^{\frac{1}{2}}H_n}{XR_n^{\frac{1}{2}}}\left(t-\frac{R_n}{c}\right)^{\frac{1}{2}}. \tag{2·18}$$

So the profile rises and falls steeply from its values at $t = R_n/c$, for each n, in approximately parabolic manner. The separate terms are bounded as $t \to \infty$, and the bound of the $(n+1)$th term in η_c is

$$-\frac{1}{\pi}\tan^{-1}\frac{2H_nX}{X^2-H_n^2} = -\frac{2}{\pi}\tan^{-1}\frac{H_n}{X}. \tag{2·19}$$

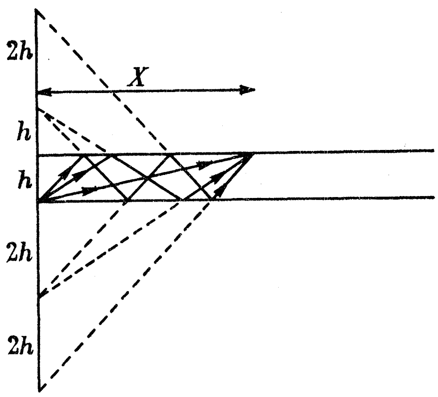

FIGURE 3. The image system for reflected arrivals.

By (2·9) the arrivals are direct ($n = 0$) and n-times reflected waves at free surface and bottom in accordance with the theory of geometrical optics, since the travel times are those from a system of images under reflexion (figure 3). The changes of sign of each term correspond to a change π of phase of the wave on reflexion at the rigid bottom.

There are two such sets of arrivals, one from each edge. For a point not over the block, with $x > a$, the first arrival is from the nearer edge, is in the same directional sense as the block upthrust, and so represents a compression wave; but the first direct arrival from the back edge is in the opposite sense (indicated by a negative sign) and can be described as a rarefaction wave. This is because the block can be split into two terms and written

$$z = H(a-x) - H(-a-x);$$

$H(a-x)$ is 0 for $x > a$ and 1 for $x < a$, so this is a positive step pushed into the liquid which one expects naturally to produce a compression wave; the term $-H(-a-x)$ is 0 for $x > -a$ and -1 for $x < -a$, which leads one to expect a rarefaction wave from this edge. At a point directly over the block, with $-a < x < a$, both first arrivals are rarefactive. Actually these are not the leading manifestations in time of the whole disturbance η, for there is also an arrival η_B from the nearest point of the block, the foot of the perpendicular to it from the observation point, which reaches the observation point first. This will be studied in § 4.

To facilitate computation, we now introduce the dimensionless time and distance quantities

$$T = ct/h, \quad Y = x/h \tag{2·20}$$

and we write also

$$N = 2n+1. \tag{2·21}$$

Then (2·17) gives

$$\eta_c = -\frac{1}{\pi} \sum_{\substack{N=1 \\ N \text{ odd}}}^{\infty} (-)^{\frac{1}{2}(N-1)} H[T-(Y^2+N^2)^{\frac{1}{2}}] \tan^{-1} \frac{2NYT(T^2-Y^2-N^2)^{\frac{1}{2}}}{T^2(Y^2-N^2)+N^2(Y^2+N^2)}. \tag{2·22}$$

We may recall that for $x > a$, we subtract the result for $X = x-a$ from that for $X = x+a$. Hence, for $X = x-a$, $Y = (x-a)/h > 0$, the leading arrival $(N = 1)$ of η_c is positive.

The first five arrivals of the purely acoustic disturbance for the particular value $Y = 1$ are shown in figure 4. The behaviour at the beginning of each arrival is well exhibited, as also the arctangent behaviour which takes over fairly soon after each steep rise or fall. A particular term in (2·22) starts at $T = (Y^2+N^2)^{\frac{1}{2}}$ and tends to $(2/\pi) \tan^{-1}(N/Y)$ in absolute value as $T \to \infty$; thus at a fixed Y, after a large number of arrivals $N \to \infty$, the oscillation tends to the value 1.

We can describe in general terms the effect of varying Y. As Y increases, the arrival time for each N increases, so that the record is stretched in the direction of the T axis. Also the limiting value $(2/\pi) \tan^{-1}(N/Y)$ of each term decreases, so that the individual peaks of the arrival record are flattened out. Ultimately, however, the oscillation still tends to 1.

The composite record of pure acoustic waves from the edges is a superposition of the record for arrivals from the nearer edge and a similarly alternating record for arrivals from the farther edge. If the observation point is over the block, the first arrivals will both be negative, in the contrary case that from the nearer edge will be positive. Moreover, after a sufficiently long time, the two sets of arrivals will interlace. The condition for the nth arrival from the farther edge to lie between the nth and $(n+1)$th arrivals from the nearer edge is $(x > 0)$

$$|x-a|^2+(2n-1)^2h^2 < (x+a)^2+(2n-1)^2h^2 < |x-a|^2+(2n+1)^2h^2.$$

The first inequality is obviously satisfied; the second gives

$$n > \tfrac{1}{2}(ax/h^2). \tag{2·23}$$

So the arrivals will interlace after the time corresponding to this value of n.

In practice, as mentioned in the Introduction, there will be considerable loss in amplitude of the reflexion arrivals, perhaps because a soft bottom will cushion the disturbance, and certainly because some of the associated energy will be transmitted into the solid medium as seismic waves. Thus the oscillation in each record will not tend to 1, but will decay to zero with time.

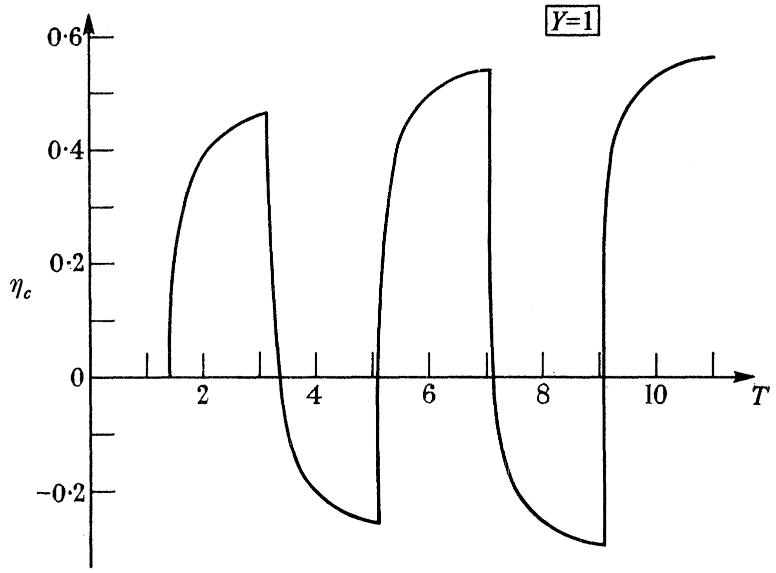

FIGURE 4. Arrivals from near edge, at a point not over the block disturbance.

It remains for us to consider the effect of gravity on the acoustic-dominant disturbance η_A. By (2·17), after some algebra, the first-order terms (involving g) are found to behave like $(t-R_n/c)^{\frac{3}{2}}$ near the arrival times, and so are less important than the terms η_c which suffer discontinuities in slope at the arrival times. Over a period of a few reflexions, $t-R_n/c$ is of order h/c if X/h is small, and of order X/c if X/h is large, so that the ratio of the perturbation terms to η_c (about $g(t-R_n/c)/c$) is gh/c^2 or gX/c^2 in the two cases. The length scale must be of the order of hundreds of kilometres for this ratio to be significant for the ocean. At much later times, however, the gravity terms will dominate; this will represent the transition zone.

By use of (2·2) we may show rigorously that the higher powers g^m or γ^m of gravity are associated with higher powers $(t-R_n/c)^{m+\frac{1}{2}}$ of the retarded time, for each integer value of m, near the $(n+1)$th arrival time. The proof first associates these powers with descending powers of p; the behaviour of the Laplace transforms for large p is connected with that of the pulses near their arrival times (see, for instance, Friedlander 1958), and this is verified directly by the remainder of the proof.

3. ASYMPTOTIC FORM OF THE EDGE ARRIVALS AT A GREAT DISTANCE

In §2 we obtained exact forms for the acoustic disturbance, and a perturbation due to gravity. But, by use of a different approach to the original integrals, we can obtain another result which is an approximation only, but should be mathematically valid when X is so large that γX becomes comparable with unity. This arises because of a curious term in the integrand of (1·12). It may possibly give a better picture of the acoustic arrivals η_A (now coupled with gravity) at large X/h than the expressions (2·17), as we have doubts about the effects of gravity on one pulse, between arrival times, for $\gamma X \sim 1$.

The approximate technique which we shall apply here is due to Sommerfeld, and depends on the branch lines in the integration with regard to k, when k (and p) are considered as complex numbers. We write

$$\sin(kX) = \tfrac{1}{2}\mathrm{i}(\mathrm{e}^{\mathrm{i}kX} - \mathrm{e}^{-\mathrm{i}kX}), \tag{3·1}$$

and take a wave expansion of the integrand as it stands, without performing any expansion in powers of γ,

$$[p^2\cosh(\mu h) + \mu g\sinh(\mu h) - \gamma(\gamma g + p^2)\sinh(\mu h)/\mu]^{-1}$$
$$= -\frac{2\mu}{\gamma(\gamma g+p^2)-\mu p^2-\mu^2 g}\sum_{n=0}^{\infty}(-)^n\left\{\frac{p^2-\mu g+\gamma(\gamma g+p^2)/\mu}{p^2+\mu g-\gamma(\gamma g+p^2)/\mu}\right\}^n \mathrm{e}^{-(2n+1)\mu h}. \tag{3·2}$$

The integration contour for $\mathrm{e}^{\mathrm{i}kX}(X > 0)$ is now distorted in the k plane into a small quadrantal arc about the origin, which is now a pole, to the positive imaginary axis, and along this axis to a large quadrantal arc connecting it to the positive real axis at infinity; on account of the factor $\mathrm{e}^{\mathrm{i}kX}$, the contribution from this last is vanishingly small (Jordan's lemma). Similarly, the integration contour for $\mathrm{e}^{-\mathrm{i}kX}$ is distorted to the negative imaginary axis and two other quadrantal arcs. The contributions from the two imaginary axis integrals are found to cancel. The contributions from the quadrants round the pole are equivalent to η_B (see §4).

Now (1·21) has no branch point as it stands, because it contains only even functions of μ. But under the wave expansion this property fails and a branch point appears at $\mu = 0$, from (1·9). When the Laplace transform is inverted, p is a complex number which can always be taken to have positive real part, but the imaginary part runs from negative to positive values. For $\mathscr{I}p > 0$, there is a branch point $\mu = 0$, $k = -\mathrm{i}(p^2/c^2+\gamma^2)^{\frac{1}{2}}$ in the fourth quadrant of the k plane, which lies between the paths for $\mathrm{e}^{-\mathrm{i}kX}$ and so will contribute to that integral; for $\mathscr{I}p < 0$, the branch point $k = \mathrm{i}(p^2/c^2+\gamma^2)^{\frac{1}{2}}$ in the first quadrant contributes to the integral for $\mathrm{e}^{\mathrm{i}kX}$.

If we take the branch cuts to be given by $\mathscr{R}\mu = 0$, so that the correct branch of μ in the k plane is $\mathscr{R}\mu \geqslant 0$, the cuts are hyperbolic arcs from these points to $\pm\mathrm{i}\infty$ respectively, and the contours must include loops about these cuts as shown in figure 5. These loops form the Sommerfeld contour (Lapwood 1949). On these loops we set

$$\mu = \mathrm{i}\nu, \tag{3·3}$$

where ν is real. Then on the corresponding branch cuts, we find to order ν^2

$$\mathrm{e}^{\mp\mathrm{i}kX} \doteqdot \exp\left[-\left(\frac{p^2}{c^2}+\gamma^2\right)^{\frac{1}{2}}X - \frac{\nu^2 X/2}{(p^2/c^2+\gamma^2)^{\frac{1}{2}}}\right]. \tag{3·4}$$

By (3·4) the principal contribution will come from the neighbourhood of the branch points $\nu = 0$ (i.e. $\mu = 0$). So the classic technique is to expand the rest of the integrand in powers of μ (or ν); integration term by term then yields the asymptotic expansion in powers of h/X, so the method is best applied for small values of this parameter. We shall keep only the leading terms here. Using (3·2), (3·3), (3·4) and (1·9), the k integral in (1·21) is transformed by the change to variable v (which runs from $-\infty$ to ∞ for $\mathscr{I}p<0$, and from ∞ to $-\infty$ for $\mathscr{I}p>0$), and eventually yields (for all p) to order $X^{-\frac{3}{2}}$

$$\eta_A \doteqdot -e^{\gamma h}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{1}{\gamma X^{\frac{3}{2}}}\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_L \frac{p\,dp\exp\left[pt-(p^2/c^2+\gamma^2)^{\frac{1}{2}}R_n\right]}{(p^2+\gamma g)\,(p^2/c^2+\gamma^2)^{\frac{1}{4}}}. \tag{3·5}$$

The integrand in (3·5) has branch point singularities at $p=\pm i\gamma c$ and simple poles at

$$p=\pm i(\gamma g)^{\frac{1}{2}}=\pm i\gamma c\surd 2.$$

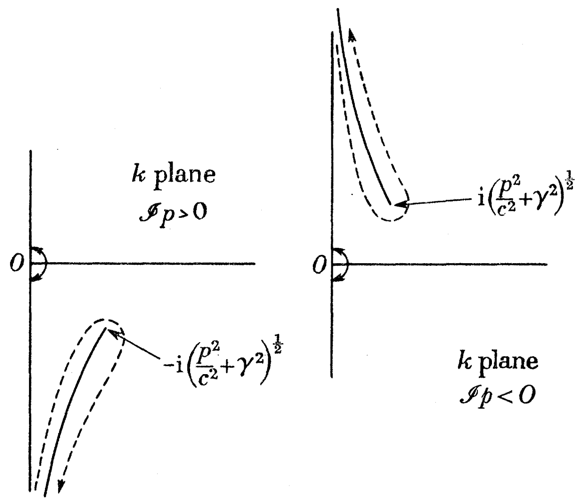

FIGURE 5. The Sommerfeld integration contour.

When the contour L is folded over the two branch lines from $p=\pm i\gamma c$ to $p=\pm i\gamma c-\infty$ for $t-R_n/c>0$, the contributions from the poles can be evaluated exactly by the calculus of residues:

$$\eta_A \doteqdot -e^{\gamma h}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(\gamma X)^{-\frac{3}{2}}\sum_{n=0}^{\infty}H\left(t-\frac{R_n}{c}\right)\cos\left[\gamma c\left(t\surd 2-\frac{R_n}{c}\right)-\frac{\pi}{4}\right]$$

$$+\text{contributions from the branch lines.} \tag{3·6}$$

These arrivals begin with an impulsive start, the value of the cosine at the start being

$$\cos\{\gamma R_n(\surd 2-1)-\tfrac{1}{4}\pi\}.$$

Thus, for large X/h or γX of order unity, a new mathematical form appears for the acoustic response, which includes both compression and gravity effects. For example, the period of a cosine is $2\pi\surd(2)c/g$ (which is of the order of 1000 sec for the ocean).

The contributions from the branch points in (3·5) can be shown to be $O(1/\gamma^2X^2)$, as against $O(1/(\gamma X)^{\frac{3}{2}})$ for (3·6), and to begin like $(t-R_n/c)^{\frac{1}{2}}$, on considering their behaviour for large p, by Friedlander's (1958) argument. Hence, for $\gamma X \gg 1$, (3·6) is a fair representation of η_A.

The result is interesting because it cannot appear for an incompressible layer for which $\gamma=0$, but apparently 'blows up' as $\gamma\to 0$. The reason is not far to seek. In the expression on the left-hand side of (3·2), the quotient $\sinh(\mu h)/\mu$ is bounded near $\mu=0$, and as $\gamma\to 0$ the term involving this quotient becomes negligible. But the wave expansion splits $\sinh(\mu h)$ into $\frac{1}{2}(e^{\mu h}-e^{-\mu h})$ and then separates the two exponentials. Hence near $\mu=0$ this term in the denominator assumes overriding importance in the power series expansion, which leads to (3·6) as shown.

We remark that if we expand in powers of γ (times a length scale), as in §2, before taking the wave expansion, thus starting from equation (2·2), we eventually obtain equation (2·18) anew for the behaviour of η_A near the arrival times, with slight differences due to the fact that the asymptotic expansion implies $X/h \gg 1$.

4. The acoustic arrivals over the block

We now turn to the contribution η_B from the small arc about $\zeta = H_n/c$ in figure 2 (§2). Substituting from (2·10) to (2·14) in (2·7) and taking the residue at $\zeta = H_n/c$, multiplied by $(-\frac{1}{2}\mathrm{i}\pi)$, and then the imaginary part as indicated for the ζ integration, we find

$$\eta_B = \mathrm{e}^{\gamma h}\frac{1}{2\pi\mathrm{i}}\int_L \frac{\mathrm{d}p}{p}\sum_{n=0}^{\infty}(-)^n\left\{1-(2n+1)\frac{g}{2pc}\right\}\exp\left\{p\left(t-\frac{H_n}{c}\right)\right\}. \tag{4·1}$$

This follows also from (2·7) at $\theta = 0$ directly, since the transformation (2·10) is regular there.

Following Jeffreys (1931), or Jeffreys & Sells (1963), we may effect a (possibly more realistic) combination of the pure acoustic terms with the variation due to gravity. For n not too large, we put

$$1-(2n+1)\frac{g}{2pc} \doteqdot \exp\left[-\frac{(2n+1)g}{2pc}\right] \tag{4·2}$$

in (4·1); then (Jeffreys & Jeffreys 1956, §21.011)

$$\eta_B = \mathrm{e}^{\gamma h}\sum_{n=0}^{\infty}(-)^n J_0\left\{\left[2(2n+1)g\left(t-\frac{H_n}{c}\right)\Big/c\right]^{\frac{1}{2}}\right\}H\left(t-\frac{H_n}{c}\right). \tag{4·3}$$

The initial behaviour of each arrival

$$\eta_B \doteqdot \mathrm{e}^{\gamma h}\sum_{n=0}^{\infty}(-)^n\left\{1-(2n+1)\frac{g}{2c}\left(t-\frac{H_n}{c}\right)\right\}H\left(t-\frac{H_n}{c}\right) \tag{4·4}$$

can be determined directly from the equation (4·1).

Now, the result (4·3), and indeed the result (4·1) from which it was derived, is independent of X. That this would happen, we could foresee since X appears only through $\mathrm{e}^{-\mathrm{i}kX}$ and when the residues at $k = 0$ ($\theta = 0$) are taken, X disappears from the calculations. Hence when $x > a$, the operation described under (1·21) means that we are to subtract two equal terms, so that η receives no contribution from η_B for points not over the block; when however $|x| < a$, we add two equal terms and obtain a contribution to η which is twice the value of (4·3).

The reason for this behaviour of η_B is as follows. For $|x| > a$, the observer is not directly over the block, and the nearest point of the block to the observer is the near edge; the farthest point is, of course, the far edge. We thus find travel times from each edge, as described in §2 with figure 3. This corresponds to the record η_A. But when the observer is directly over the block, the nearest point of the block is the foot of the perpendicular to it from the observer (figure 6), and since this perpendicular has length h the first arrival, for which

$n = 0$, comes in at time $t = h/c$. The wave is then reflected at the surface and returns to the bottom, at the point from which it started. The amplitude of the disturbance has been assumed small compared with the typical length scale of the system, so the return travel path is also of length h; the wave suffers a further reflexion at the bottom and again travels up to the free surface, reaching it at time $3h/c$, and the process is repeated.

This record η_B is superposed on the two records from the edge discussed in the preceding section, for the case $|x| < a$.

By (4·3) or (4·4), each arrival begins with an impulsive start, and since $\eta = 2\eta_B +$ terms (η_A), the jump is twice the size of the initial disturbance. This would explain the violent shock felt by an ocean vessel which happened to pass over a submarine volcano at the moment of explosion, as reported by Unoki & Nakano (1953). The subsequent shocks would not be so severe, because the reflexions at the sea bottom are subject to attenuation (in the same way as the reflexions η_A), and also because a volcano, being more triangular than block in shape, would reflect the rays returning from the surface in a different direction.

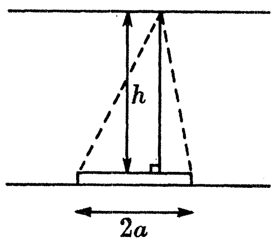

FIGURE 6. Direct arrival paths at point over the block.

The factor 2 arises because η_B is made up from both the incident wave and the reflected wave. The alternations in sign are due to the change π of phase in the waves on reflexion at the rigid bottom. The pure acoustic effect associated with η_B resembles that of a pipe, open at one end and with a piston at the other end, when the piston is suddenly displaced a short distance along the tube.

As in the working for the disturbance η_c, we may substitute $T = ct/h$, from (2·20), and write the *whole* contribution to η from (4·3) as

$$2\,[H(x+a)-H(x-a)]\,e^{\gamma h}\sum_{n=0}^{\infty}(-)^n J_0\left\{\left[2(2n+1)\frac{gh}{c^2}(T-2n-1)\right]^{\frac{1}{2}}\right\}H(T-2n-1)$$
$$= 2[H(x+a)-H(x-a)]\,e^{\gamma h}\,\eta_b. \quad (4·5)$$

The Heaviside operators in x give the block locality, and the factors $2e^{\gamma h}$ are left out of the definition of η_b for convenience.

For the comparatively large value $gh/c^2 = \frac{1}{50}$, and $0 \leqslant T \leqslant 25$, the behaviour of η_b is depicted in figure 7. The first six arrivals or so are nearly straight line portions following the arrival times, in agreement with (4·4), the later terms show concavity upwards until the twelfth $(23 \leqslant T \leqslant 25)$ which begins to curve downwards; simultaneously, the rise of the initial value of the *odd* swings (first, third, ...) is halted. Later, the oscillatory nature of the Bessel functions will cause the arrivals to fluctuate after the impulse; this begins earlier or later according as the value of gh/c^2 is larger or smaller.

We observe that $\eta_b = 0$ when $T = 4m$, where m is an integer; for (4·5) gives

$$\eta_b \Big|_{T=4m} = \sum_{n=0}^{2m-1} (-)^n J_0 \left[\left\{ 2(2n+1)(4m-2n-1)\frac{gh}{c^2} \right\}^{\frac{1}{2}} \right], \tag{4·6}$$

and the terms $\sum_{n=0}^{m-1}$ and $\sum_{2m-1}^{m}$ cancel in pairs.

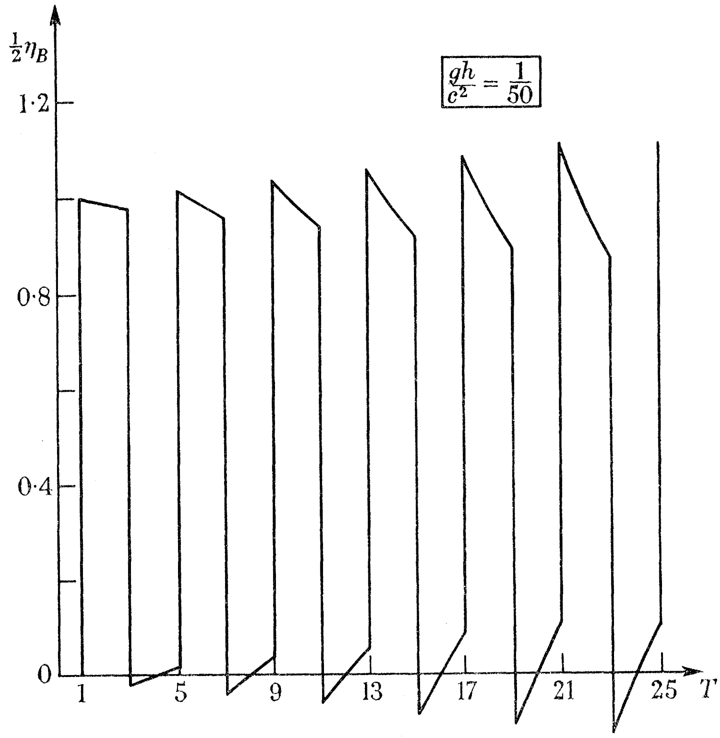

FIGURE 7. Jump arrivals η_B over the block.

By treating (1·21) in a different way, we can obtain some information about the cumulative behaviour of the terms in η_b for large times. Removing the factors

$$2\,e^{\gamma h}[H(x+a) - H(x-a)],$$

and neglecting γ^2, but not performing the wave expansion, we find that the residue at $k = 0$ ($\theta = 0$) gives from (1·21)

$$\eta_b = \frac{1}{2\pi i}\int_L \frac{\frac{1}{2}e^{pt}\,dp}{p\cosh(ph/c) + \gamma c \sinh(ph/c)}, \tag{4·7}$$

g has been replaced by $2\gamma c^2$.

If we expand in powers of $e^{-2ph/c}$, we will obtain (4·4) anew. But instead of this we may evaluate (4·7) approximately by residue calculus. The integrand has an infinity of simple poles on the imaginary p axis, and also at $p = 0$; each of these will contribute a residue when the contour L is closed by a large semicircle on the left for $t > h/c$.

When $p \neq 0$, replace p by $i\omega$ where ω is real; then the poles occur where

$$\cot\left(\frac{\omega h}{c}\right) = -(\gamma h)\left(\frac{c}{\omega h}\right), \tag{4·8}$$

of which an approximate solution is

$$\frac{\omega h}{c} = (N+\tfrac{1}{2})\,\pi + \frac{\gamma h}{(N+\tfrac{1}{2})\,\pi} \tag{4·9}$$

for integer values of N. If $\gamma h \ll 1$, this is a good approximation even for $N = 0$; it is certainly good for sufficiently large N.

We note that if we had taken higher powers of γ, they would have yielded higher powers of (γh) on the right-hand side of (4·9).

At the poles $p = i\omega$, with ω given by (4·9),

$$\frac{\mathrm{d}}{\mathrm{d}p}\left\{p\cosh\left(\frac{ph}{c}\right)+\gamma c\sinh\left(\frac{ph}{c}\right)\right\} \doteqdot (-)^{N+1}\left\{(N+\tfrac{1}{2})\,\pi+\frac{2\gamma h}{(N+\tfrac{1}{2})\pi}\right\}. \tag{4·10}$$

So, pairing $N = 0, 1, 2, \ldots$ with $N = -1, -2, -3, \ldots$, we find for η_b in (4·7) the value

$$\eta_b \doteqdot H\left(t-\frac{h}{c}\right)\left[\sum_{N=0}^{\infty}(-)^{N+1}\frac{\cos\{(N+\frac{1}{2})\,\pi+\gamma h/(N+\frac{1}{2})\,\pi\}T}{(N+\frac{1}{2})\,\pi+2\gamma h/(N+\frac{1}{2})\,\pi}+\frac{\frac{1}{2}}{1+\gamma h}\right]. \tag{4·11}$$

The last term is the residue at $p = 0$. We have written $T = ct/h$ as before.

We look next for an approximate form of (4·11) for large T. Although, as it happens, the sum can be found exactly for $0 \leqslant T < 1$, by writing z for $(N+\frac{1}{2})\,\pi$, inserting a factor $(\sec z)$ and integrating round a contour which first encloses the positive integers and is afterwards suitably distorted to give the residue at the other pole, the method breaks down for $T > 1$ because the distortion cannot be performed. For the same reason, the steepest descent technique fails with the new integral.

A different way out is therefore sought for large T. We apply the Euler-Maclaurin approximation to (4·11), replacing the sum by an integral over the range $-\infty$ to ∞; we may also change the variable to z, where

$$z = (N+\tfrac{1}{2})\,\pi. \tag{4·12}$$

Then

$$\eta_b = \frac{1}{2\pi}\,\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi}\int_{-\infty}^{\infty}\exp\left[\mathrm{i}T\left(z+\frac{\gamma h}{z}\right)-\ln\left(z+\frac{2\gamma h}{z}\right)+\mathrm{i}z\right]\mathrm{d}z+\frac{\frac{1}{2}}{1+\gamma h}. \tag{4·13}$$

For values of γh and T with $\gamma h \ll 1$, $(\gamma h)^{\frac{1}{2}}\,T \gg 1$, the integral can be evaluated by the method of steepest descent. There are two saddle points, at

$$z = (\gamma h)^{\frac{1}{2}}\left\{\pm 1+\frac{\mathrm{i}}{6(\gamma h)^{\frac{1}{2}}\,T}\right\}, \tag{4·14}$$

and they give (substituting for T)

$$\eta_b = \frac{1}{3\pi^{\frac{1}{2}}}\left(\frac{2h}{g}\right)^{\frac{1}{4}}t^{-\frac{1}{2}}\sin\left\{\left(\frac{2g}{h}\right)^{\frac{1}{2}}t-\frac{3\pi}{4}\right\}+\frac{\frac{1}{2}}{1+\gamma h}. \tag{4·15}$$

Thus as time goes on, this component oscillates about the mean height 1 of the pure acoustic impulses, since $\eta_B = 2\,\mathrm{e}^{\gamma h}\eta_b$ in $|x| < a$. The period of the motion is $\pi(2h/g)^{\frac{1}{2}}$ and the amplitude decays as $t^{-\frac{1}{2}}$.

We note that the effect of compressibility, manifest through the sound velocity c, no longer appears for this part of the disturbance (this being not the whole but only a component that is expected in $|x| < a$ together with η_A); the gravity wave dominates and we are in the last stage of the transition zone between the two governing phenomena.

5. The initial surface displacement in the incompressible limit

Before we turn to the regions in space and time where the controlling mechanism is that due to gravity, with compressibility playing only the part of a perturbation of the motion, there is one aspect of the purely acoustic disturbance, which deserves a glance. Let us consider what happens to the sum total of this disturbance when $c \to \infty$ and $t \to 0$ in such a way that $ct \to \infty$, so that all the arrivals which make up η_A and η_B come in together in an infinitesimally short time after the initial block disturbance. The sum total in this limit must be the same as that obtained by taking an incompressible fluid in (1·21) and calculating the initial elevation of the surface over the block; that is, this initial elevation can indeed be regarded as a pure acoustic effect.

First, let us consider η_A as given by (2·17).

In the limit we are considering, the gravity terms contribute nothing; for the first two have $t(\to 0)$ as a factor, and the last one has $1/c^2\ (\to 0)$. So only the acoustic part of η_A will give any contribution. We have

$$\tan^{-1}\frac{2H_n X ct(c^2t^2-R_n^2)^{\frac{1}{2}}}{c^2t^2(X^2-H_n^2)+R_n^2 H_n^2} = 2\tan^{-1}\frac{H_n}{X} - \frac{H_n X}{(ct)^2} + O\left\{\left(\frac{X}{ct}\right)^4\right\}. \tag{5·1}$$

Hence, as $ct \to \infty$, (2·17) gives (compare (2·19))

$$\eta_A \to -\frac{2}{\pi}\sum_{n=0}^{\infty}(-)^n \tan^{-1}\frac{(2n+1)\,h}{X}, \tag{5·2}$$

since $e^{\gamma h} \to 1$; we replace H_n by $(2n+1)\,h$.

Now, if

$$S \equiv S(w) = \sum_{n=0}^{\infty}(-)^n \tan^{-1}(2n+1)\,w, \tag{5·3}$$

then

$$\frac{ds}{dw} = \sum_{n=0}^{\infty}(-)^n \frac{2n+1}{1+(2n+1)^2 w^2} = \frac{\pi}{4w^2\cosh(\pi/2w)} \tag{5·4}$$

(by contour integration) is uniformly valid in any region not including $w = 0$. To recover S, we now require its value for some w. Taking $w = \infty$, all the terms in (5·3) are numerically equal to $\frac{1}{2}\pi$, the sum oscillates between the limits 0 and $\frac{1}{2}\pi$ with equal weights, and may be taken to have its mean value $\frac{1}{4}\pi$ (Cesarò convention). Taking $w > 0$, and making a change of variable, we have

$$S = \tfrac{1}{4}\pi - \tfrac{1}{4}\pi\int_w^{\infty}\frac{dw'}{w'^2\cosh(\pi/2w')} = \tfrac{1}{4}\pi - \tfrac{1}{2}\int_0^{\pi/2w}\frac{dw''}{\cosh w''} = \tan^{-1}\left\{\exp\left(-\frac{\pi}{2w}\right)\right\}. \tag{5·5}$$

We observe that as $w \to 0$ through positive values, S given by either (5·3) or (5·5) tends to 0 also. When $w < 0$, we find similarly

$$S = -\tan^{-1}\{\exp(\pi/2w)\}. \tag{5·6}$$

So S is indeed an odd function of w, as one expects.

Now, let us put $w = h/X$ and see what η_A contributes to η when two values of η_X are compounded as described under (1·21). First, let us take $x > a$. Then $(x-a)$ and $(x+a)$ are both positive, and subtracting the value for $X = x-a$ from that for $X = x+a$, we obtain (continuing to write η_A for this part of the disturbance)

$$\eta_A = \frac{2}{\pi}\left[\tan^{-1}\exp\left\{-\frac{\pi(x-a)}{2h}\right\} - \tan^{-1}\exp\left\{-\frac{\pi(x+a)}{2h}\right\}\right] = \frac{2}{\pi}\tan^{-1}\frac{\sinh(\pi a/2h)}{\cosh(\pi x/2h)} \quad (|x| > a). \tag{5·7}$$

For $|x| < a$, the results for $X = a - x$ and $X = a + x$ are added, and we find here

$$\eta_A = -\frac{2}{\pi}\tan^{-1}\frac{\cosh(\pi x/2h)}{\sinh(\pi a/2h)} \quad (|x| < a). \tag{5·8}$$

We now turn to the contributions η_B which give the arrivals over the block as in §4. For $|x| > a$, we are not directly over the block, and the contribution is

$$\eta_B = 0 \quad (|x| > a). \tag{5·9}$$

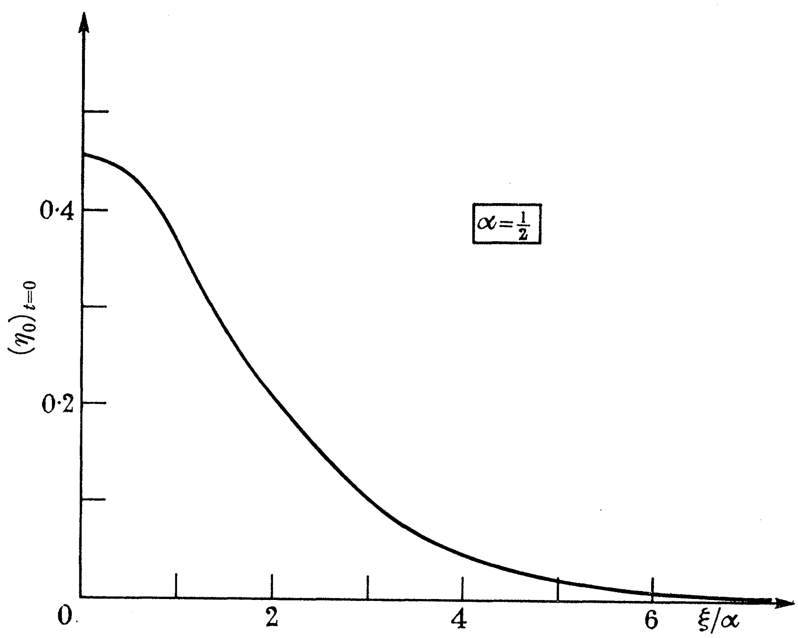

FIGURE 8. The initial elevation (incompressible fluid).

For $|x| < a$, we will obtain values for η_B twice those given by (4·3) or (4·4). Proceeding to the limits $c \to \infty$, $t \to 0$, with $ct \to \infty$, we again find that the gravity correction disappears so that the initial disturbance is still pure acoustic, and the oscillatory sum results:

$$\eta_B = 2\sum_{n=0}^{\infty}(-)^n = 2 - 2 + 2 - 2 + \dots.$$

We use the same convention as before and take the value of the sum as 1, the mean of the fluctuations. This is physically sound, for we can use the pipe analogy discussed in §4, and argue that the unit movement of the piston at one end induces unit movement at the other end, if the pipe is filled with an incompressible block of fluid or matter. So

$$\eta_B = 1 \quad (|x| < a). \tag{5·10}$$

Then

$$\eta = \eta_A + \eta_B$$

$$= -\frac{2}{\pi}\tan^{-1}\frac{\cosh(\pi x/2h)}{\sinh(\pi a/2h)} + 1 = \frac{2}{\pi}\tan^{-1}\frac{\sinh(\pi a/2h)}{\cosh(\pi x/2h)} \quad (|x| < a) \tag{5·11}$$

by (5·8) and (5·10); and the same result follows from (5·7) and (5·9) for $|x| > a$. So, for all x, the initial elevation in incompressible fluid is

$$\eta_0 = \frac{2}{\pi}\tan^{-1}\frac{\sinh(\pi a/2h)}{\cosh(\pi x/2h)}. \tag{5·12}$$

The graph of η_0 as a function of $x/h = \xi$ is shown in figure 8 for the value $a/h = \alpha = \frac{1}{2}$.

The function η_0 has the expected properties of being symmetric about the axis $x = 0$, with a maximum at that point, of diminishing to zero as $x \to \infty$ and as $a/h \to 0$ and approaching the value 1 as $a \to \infty$ for $|x| < a$. It is, of course, also a smooth curve. This property is a little difficult to foresee from the start, as the pipe effect might be hastily assumed to be the only contribution (Webb 1962).

6. The development of the gravity wave

We proceed now to examine the form of the disturbance η when gravity is the dominant mechanism, as in the classical theory of waves in incompressible media. To begin this investigation, and also to include to first-order the effects of compressibility on the motion, we expand the integrand of (1·21) in powers of each of the two quantities $(p/kc)^2$, g/kc^2, and neglect squares and products thereof. It is convenient to write $p = i\omega$, so that

$$\mu^2 = k^2 + \gamma^2 - \frac{\omega^2}{c^2} \doteqdot k^2 - \frac{\omega^2}{c^2}, \tag{6·1}$$

and to the first-order

$$\eta_X = e^{\gamma h} \frac{1}{2\pi^2 i} \int_\Omega \omega \, d\omega \, e^{i\omega t} \int_0^\infty \frac{dk}{k} \frac{\sin(kX)}{\omega^2 \cosh(\mu h) - \mu g \sinh(\mu h) + \gamma(\gamma g - \omega^2) \sinh(\mu h)/\mu}$$

$$= e^{\gamma h} \frac{1}{\pi} \int_0^\infty \frac{dk}{k} \sin(kX) \frac{1}{2\pi i} \int_\Omega \frac{\omega \, d\omega \, e^{i\omega t}}{\omega^2 \cosh(kh) - gk \sinh(kh)}$$
$$\times \left\{1 + \frac{1}{2} \frac{\omega^2}{k^2 c^2} (kh) \frac{\omega^2 \sinh(kh) - gk \cosh(kh)}{\omega^2 \cosh(kh) - gk \sinh(kh)}\right\}. \tag{6·2}$$

Here Ω is a contour in the ω plane which runs from $-\infty$ to ∞ below all singularities of the integrand. In (6·2) these singularities are poles, and the integration with regard to ω can be performed by the calculus of residues. For $t < 0$, we complete the contour Ω with a large semicircle in the lower half-plane, and since Ω has no singularities below it the integral is zero. For $t > 0$, Ω is closed by a semicircle in the upper half-plane; the first term in the integrand of (6·2) has simple poles at $\omega = \pm\{gk \tanh(kh)\}^{\frac{1}{2}}$, and these contribute together

$$e^{\gamma h} H(t) \frac{1}{\pi} \int_0^\infty \frac{dk}{k} \frac{\sin(kX)}{\cosh(kh)} \cos\{gk \tanh(kh)\}^{\frac{1}{2}} t. \tag{6·3}$$

The poles are the values of the frequency of simple harmonic gravity waves in water of depth h.

The second term (first-order approximation) has double poles, and after some algebra the residues are found to give for this term

$$e^{\gamma h} H(t) \frac{1}{2\pi} \frac{gh}{c^2} \int_0^\infty \frac{dk}{k} \frac{\sin(kX)}{\cosh(kh)} [\{2 \tanh^2(kh) - 1\} \cos\{gk \tanh(kh)\}^{\frac{1}{2}} t$$
$$+ \tfrac{1}{2}\{gk \tanh(kh)\}^{\frac{1}{2}} t \operatorname{sech}^2(kh) \sin\{gk \tanh(kh)^{\frac{1}{2}}\} t]. \tag{6·4}$$

We substitute in terms of dimensionless parameters and variable:

$$kh = \lambda, \quad x/h = \xi, \quad a/h = \alpha, \quad X/h = \beta, \quad (g/h)^{\frac{1}{2}} t = \tau. \tag{6·5}$$

Then, combining (6·3) and (6·4)

$$\eta_X = \frac{1}{\pi}\mathrm{e}^{\gamma h} H(\tau)\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda}\frac{\sin(\beta\lambda)}{\cosh\lambda}\Big\{\cos(\lambda\tanh\lambda)^{\frac{1}{2}}\tau + \frac{1}{2}\frac{gh}{c^2}[(2\tanh^2\lambda-1)\cos(\lambda\tanh\lambda)^{\frac{1}{2}}\tau + \tfrac{1}{2}(\lambda\tanh\lambda)^{\frac{1}{2}}\tau\,\mathrm{sech}^2\lambda\sin(\lambda\tanh\lambda)^{\frac{1}{2}}\tau]\Big\}. \quad (6\cdot6)$$

We now see that the velocity scale implicitly assumed small compared with c in the expansion leading to (6·2) is $(gh)^{\frac{1}{2}}$, the velocity of waves long compared with the depth. This is exactly the same as the assumption $\gamma h \ll 1$ in the discussion of the acoustic waves, and holds equally good for terrestrial oceans.

Let us write an expansion

$$\eta_X = H(\tau)\sum_{n=0}^{\infty}\bar{\eta}_n(\beta,\tau)\left(\frac{gh}{c^2}\right)^n = H(\tau)\left(\bar{\eta}_0+\bar{\eta}_1\frac{gh}{c^2}+\dots\right) \quad (6\cdot7)$$

and we approximate here also to

$$\mathrm{e}^{\gamma h} = \exp\left(\frac{gh}{2c^2}\right) = 1+\frac{gh}{2c^2}+\dots. \quad (6\cdot8)$$

Combining (6·6) and (6·8), we find

$$\bar{\eta}_0 = \bar{\eta}_0(\beta,\tau) = \frac{1}{\pi}\int_0^\infty\frac{\mathrm{d}\lambda}{\lambda}\frac{\sin(\beta\lambda)}{\cosh\lambda}\cos(\lambda\tanh\lambda)^{\frac{1}{2}}\tau, \quad (6\cdot9)$$

$$\bar{\eta}_1 = \bar{\eta}_1(\beta,\tau) = \frac{1}{\pi}\int_0^\infty\frac{\mathrm{d}\lambda}{\lambda}\frac{\sin(\beta\lambda)}{\cosh\lambda}\{\tanh^2\lambda\cos(\lambda\tanh\lambda)^{\frac{1}{2}}\tau + \tfrac{1}{4}\mathrm{sech}^2\lambda\,(\lambda\tanh\lambda)^{\frac{1}{2}}\tau\sin(\lambda\tanh\lambda)^{\frac{1}{2}}\tau\}. \quad (6\cdot10)$$

In this section, we concern ourselves only with the initial behaviour of the gravity wave, that is, the initial surface elevation and its spreading out to either side. We may expand the integrands in powers of τ^2 and integrate the first few terms exactly, while it will appear that, in principle, the whole series could be integrated. Three terms will suffice to display the generation of the first wave from over the block.

The principal contribution to the integral, under expansion, comes from the neighbourhood of the origin, $\beta\lambda = O(1)$, so that the expansion will be valid for $\tau^2/\beta \ll 1$ or $gt^2/X \ll 1$; in order for a few terms in the series to give a good approximation we therefore require that gt^2/a or gt^2/x shall be small compared with 1, according to whether we are directly over the block or not. At $X = 0$ the method possibly breaks down; but we assume physical continuity of the surface over the edges of the block, since $\eta_X = 0$ when $X = 0$. Then (6·9) becomes

$$\bar{\eta}_0 = \frac{1}{\pi}\int_0^\infty\frac{\mathrm{d}\lambda}{\lambda}\frac{\sin(\beta\lambda)}{\cosh\lambda}\sum_{s=0}^{\infty}\frac{(-)^s}{(2s)!}(\tau^2\lambda\tanh\lambda)^s = \frac{1}{\pi}\sum_{s=0}^{\infty}\frac{(-)^s}{(2s)!}\tau^{2s}I_{s-1,s}, \quad (6\cdot11)$$

where
$$I_{m,n} = \int_0^\infty\frac{\sin(\beta\lambda)}{\cosh\lambda}\lambda^m\tanh^n\lambda\,\mathrm{d}\lambda. \quad (6\cdot12)$$

For $s = 0$, we have $m = -1$, $n = 0$, and

$$I_{-1,0} = \int_0^\infty\frac{\sin(\beta\lambda)}{\cosh\lambda}\frac{d\lambda}{\lambda} = \tfrac{1}{2}\mathscr{I}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\beta\lambda}}{\cosh\lambda}\frac{\mathrm{d}\lambda}{\lambda}. \quad (6\cdot13)$$

For $\beta > 0$ the contour in the complex λ plane can be completed by a large semicircle in the upper half-plane passing between two of the points $\lambda = (N+\frac{1}{2})\pi i$, where N is a non-negative integer, which are zeros of $\cosh\lambda$, and also a small semicircle over the origin. The residues at the poles give on summation

$$I_{-1,0} = \tfrac{1}{2}\pi - 2\tan^{-1}\exp(-\tfrac{1}{2}\pi\beta) \quad (\beta > 0). \tag{6·14}$$

Similarly
$$I_{-1,0} = -\{\tfrac{1}{2}\pi - 2\tan^{-1}\exp(\tfrac{1}{2}\pi\beta)\} \quad (\beta < 0). \tag{6·15}$$

To find η from η_X we have to subtract the result for $X = x-a$ from that for $X = x+a$, or (which in this case is the same thing) add the result for $X = a-x$ to that for $X = a+x$. The results hold for both $|x| \gtrless a$, since $I_{-1,0}$ (and in general, $I_{m,n}$) is an odd function of $\beta = X/h$. So, when $\bar{\eta}_0$ has been found, we add its values for $\beta = \alpha-\xi$ and $\beta = \alpha+\xi$, α and ξ being defined by (6·5), and this gives η_0, the correct surface elevation over the block at time t. Likewise for $\bar{\eta}_1, \bar{\eta}_2, \ldots$. The analogue of (6·7) is

$$\eta = H(\tau)\sum_{n=0}^{\infty}\left(\frac{gh}{c^2}\right)^n \sum_{\beta=\alpha\pm\xi}\bar{\eta}_n(\beta,\tau) = H(\tau)\sum_{n=0}^{\infty}\eta_n(\alpha,\xi,\tau)\left(\frac{gh}{c^2}\right)^n. \tag{6·16}$$

From (6·11), (6·14) and the above, the value of $\eta_0(\alpha,\xi,0)$ is

$$1-\frac{2}{\pi}[\tan^{-1}\exp\{-\tfrac{1}{2}\pi(\alpha+\xi)\}+\tan^{-1}\exp\{-\tfrac{1}{2}\pi(\alpha-\xi)\}] = \frac{2}{\pi}\tan^{-1}\frac{\sinh(\tfrac{1}{2}\pi\alpha)}{\cosh(\tfrac{1}{2}\pi\xi)}; \tag{6·17}$$

the same result is obtained if $\xi > \alpha$ and the other results are used, as a check.

This value of $\eta_0(\alpha,\xi,0)$, the initial elevation of incompressible fluid, is in complete agreement with the expression (5·12) found by adding all the pure acoustic arrivals in the limit $c\to\infty$. Thus $\eta_0(\alpha,\xi,0)$ can be looked upon either as this acoustic limit or as the effect of the block displacing a volume of fluid upwards and to one side.

As remarked in § 5, and shown by figure 8, the function $\eta_0(\alpha,\xi,0)$ exhibits the properties of taking a maximum value at $\xi = 0$, about which it is symmetric, and of decaying to zero at large ξ.

The further terms in $\bar{\eta}_0$(and hence η_0), which represent the spreading out of the initial elevation into waves, are given by (6·11) in the form of more complicated integrals. For $s > 0$, the pole at the origin disappears but multiple poles appear at the zeros of $\cosh\lambda$ and the residues become awkward to determine and to sum. We therefore apply other methods. First, we need $I_{0,1}$. Introduce a new parameter K into $I_{-1,0}$, and we obtain from (6·14), (6·15)

$$\int_0^\infty \frac{d\lambda}{\lambda}\frac{\sin(\beta\lambda)}{\cosh(\lambda K)} = (\operatorname{sgn}\beta)\left\{\tfrac{1}{2}\pi - 2\tan^{-1}\exp\left(-\frac{\pi|\beta|}{2K}\right)\right\} \quad (K>0). \tag{6·18}$$

Differentiate both sides of (6·18) with respect to K and then put $K = 1$. Then

$$I_{0,1} = \frac{\tfrac{1}{2}\pi\beta}{\cosh(\tfrac{1}{2}\pi\beta)}. \tag{6·19}$$

This gives the next term in (6·11) for $\bar{\eta}_0$, and hence η_0. The new term in η_0 is found to be positive for ξ rather larger than α, and negative when ξ is in the neighbourhood of zero, so that at first the profile falls in the centre and rises farther out from the block; this is the first stage in the development of the wave motion.

To find more terms in principle, we return to the general expression (6·12) for $I_{m,n}$. Integrating by parts twice and rearranging, we have for $m, n \geqslant 0$

$$(n+1)(n+2)I_{m,n+2} = 2m(n+1)I_{m-1,n+1} + \{n^2+(n+1)^2-\beta^2\}I_{m,n} - m(m-1)I_{m-2,n} - 2mnI_{m-1,n-1} - n(n-1)I_{m,n-2}. \tag{6·20}$$

The recurrence relation (6·20) serves to reduce the suffix n to 0 or to 1, and we can obtain the further terms in (6·11) successively. It is convenient to keep η_0 in the form of a sum over two values of β; to order τ^6, we find

$$\eta_0 = \frac{2}{\pi}\tan^{-1}\frac{\sinh(\frac{1}{2}\alpha\pi)}{\cosh(\frac{1}{2}\xi\pi)} - \tfrac{1}{4}\tau^2 \sum_{\beta=\alpha\pm\xi} \frac{\beta}{\cosh(\frac{1}{2}\pi\beta)}$$
$$+\frac{1}{4!\,2}\tau^4 \sum_{\beta=\alpha\pm\xi}\left\{\frac{\beta}{\cosh(\frac{1}{2}\pi\beta)} + \tfrac{1}{4}\pi(1-\beta^2)\frac{\tanh(\frac{1}{2}\pi\beta)}{\cosh(\frac{1}{2}\pi\beta)}\right\}$$
$$-\frac{1}{6!\,2}\tau^6 \sum_{\beta=\alpha\pm\xi}\left[\frac{\beta}{\cosh(\frac{1}{2}\pi\beta)} + \tfrac{1}{2}\pi(\tfrac{5}{3}-\beta^2)\frac{\tanh(\frac{1}{2}\pi\beta)}{\cosh(\frac{1}{2}\pi\beta)} - \frac{\frac{1}{48}\pi^2\beta(5-\beta^2)\{\cosh(\pi\beta)-3\}}{\cosh^3(\frac{1}{2}\pi\beta)}\right] \quad (6·21)$$

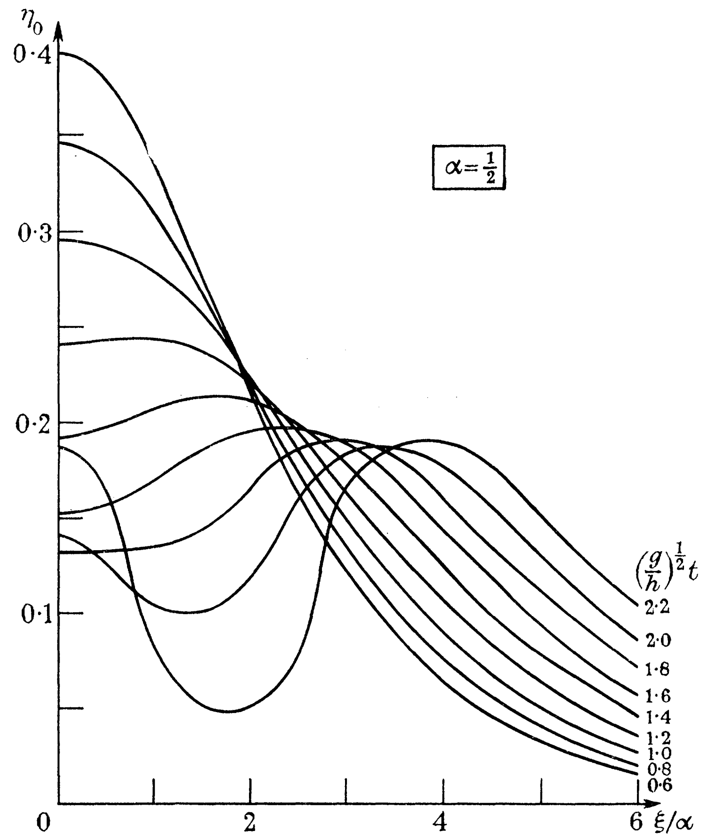

Figure 9. Development of first wave (incompressible fluid).

For $\alpha = \frac{1}{2}$, that is $2a = h$ or the block width equal to the liquid layer depth, the development of the first wave in incompressible fluid is shown in figure 9. Three terms of the series (6·21) were taken, and the wave is seen to be fully developed when $\tau = (g/h)^{\frac{1}{2}}t = 2·2$; after this time, the fourth term (which is negative near $\xi = 0$) becomes important and forces the water surface down again near the origin, to start the next wave. In detail, the hump which is formed originally at the origin falls at first and begins to separate into two wave crests a little before $\tau = 1·2$, one in $x > 0$ and the other in $x < 0$; these both travel outwards (only $x > 0$ is shown in the figures), and the trough in each develops at about $\tau = 1·8$ and follows the crest outwards. When the development of the first wave is complete, at $\tau = 2·2$, the amplitude of the wave is 0·07 times the bottom initial disturbance, and the wavelength is about twice the layer depth h.

We now turn to the modification of η_0 by compressibility, as exemplified by (6·7) and particularly the first term $\bar{\eta}_1$, given by (6·10). The expansion in powers of τ gives

$$\bar{\eta}_1 = \frac{1}{2\pi}\sum_{s=0}^{\infty}\frac{(-)^s}{(2s)!}\tau^{2s}\{(2+s)\,I_{s-1,\,s+2}-sI_{s-1,\,s}\}. \tag{6·22}$$

The integral $I_{-1,\,2}$ (corresponding to $s=0$) cannot be expressed in terms of elementary functions, but it can be manipulated into a series akin to that for Euler's dilogarithmic function, or expressed as a finite integral. The other integrals are all calculable in principle, and after much algebra we find to order τ^6

$$\begin{aligned}\eta_1 = \sum_{\beta=\alpha\pm\xi}&\left\{\frac{1}{\pi}(\beta^2-1)\tan^{-1}\exp\left(-\tfrac{1}{2}\pi|\beta|\right)-\frac{8}{\pi^3}\int_0^{\exp(-\frac{1}{2}\pi|\beta|)}(\tan^{-1}\rho)\,(\ln\rho)\,\frac{\mathrm{d}\rho}{\rho}\right\}\operatorname{sgn}\beta\\ &-\frac{\tau^2}{2!}\sum_{\beta=\alpha\pm\xi}\frac{\beta(3-\beta^2)}{8\cosh\left(\frac{1}{2}\pi\beta\right)}+\frac{\tau^4}{4!}\frac{1}{48}\sum_{\beta=\alpha\pm\xi}\frac{8\beta(4-\beta^2)+\pi(3-8\beta^2+\beta^4)\tanh\left(\frac{1}{2}\pi\beta\right)}{\cosh\left(\frac{1}{2}\pi\beta\right)}\\ &-\frac{\tau^6}{6!}\sum_{\beta=\alpha\pm\xi}\left\{\frac{1}{24}\frac{\beta(27-5\beta^2)}{\cosh\left(\frac{1}{2}\pi\beta\right)}+\frac{\pi}{96}\frac{(5\beta^4-54\beta^2+29)\tanh\left(\frac{1}{2}\pi\beta\right)}{\cosh\left(\frac{1}{2}\pi\beta\right)}\right.\\ &\qquad\left.-\frac{\pi^2}{768}\frac{\beta(\beta^4-18\beta^2+29)\{\cosh(\pi\beta)-3\}}{\cosh^3\left(\frac{1}{2}\pi\beta\right)}\right\}\end{aligned} \tag{6·23}$$

No convenient law seems to follow for the successive terms in either η_0 or η_1, and for further terms, although they can be computed in principle, the labour of computation increases.

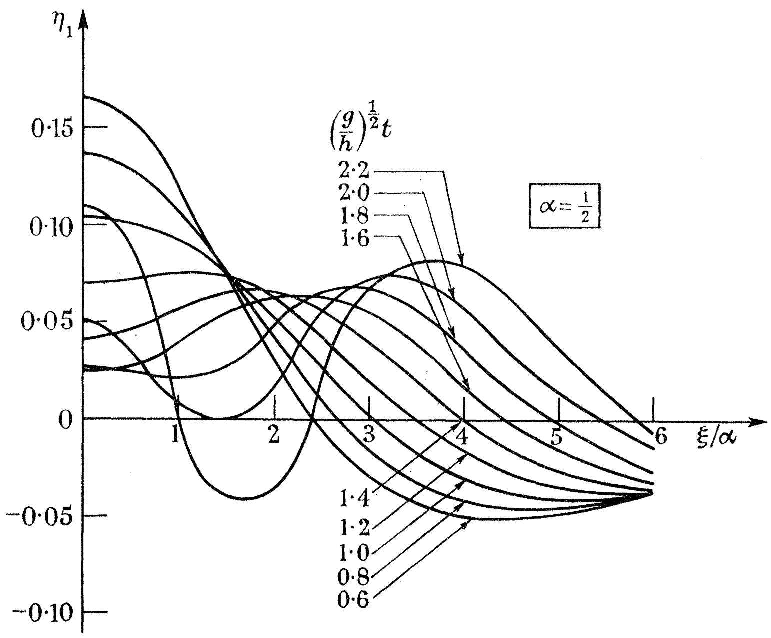

FIGURE 10. The 'compressibility wave' to first order.

For the same value of α as before, $\alpha=\frac{1}{2}$, the curves for η_1 are shown in figure 10. As the time taken for the first acoustic pulse in compressible fluid to reach the surface is at least h/c, and (following the argument of §5) many such pulses will have to arrive before the surface is effectively smoothed out, we require values of $(g/h)^{\frac{1}{2}}t$ considerably larger than $(gh/c^2)^{\frac{1}{2}}$. For an ocean 2·5 km deep, with $c = 15{\cdot}2\times10^4$ cm/s gh/c^2 is about 0·01, so the first time shown, $\tau = 0{\cdot}6$, is fairly plausible, and the development of η_1 is continued from this time. There is a depression on each side of $\xi = 0$, which travels outwards and releases a crest and trough in a manner similar to η_0.

Since the scales of the two figures 9 and 10 are in the ratio $gh/c^2 \doteqdot 0{\cdot}01$, we see that the average difference between η_0 and $\eta \doteqdot \eta_0+\eta_1 gh/c^2$ is about $1/200\,\eta_0$, so that in the development of the first gravity wave over the block, the effect of compressibility is hardly noticeable. It can be easily visualized, for two corresponding curves η_0, η_1 at given τ, ξ and α, as a very slight perturbation or bulge, on one side or the other according to the sign of η_1, of the η_0 curve.

7. The asymptotic behaviour of the leading gravity wave at large times

We again consider the equation (6·6) which gives η_x as a gravity wave with a compression perturbation term. We obtain η from η_x by putting $\beta = \alpha+\xi, \alpha-\xi$, and adding. Let us perform this operation and then make a change also in the definitions of η_0, η_1, of equation (6·16) ff.; keep $e^{\gamma h}$ as a constant multiplying factor representing the effect of decreasing density as we rise from the layer bed, and write instead of (6·16)

$$\eta = e^{\gamma h}\left(\eta_0+\frac{gh}{c^2}\eta_1+\dots\right). \tag{7·1}$$

Then, by (6·6)

$$\eta_0 = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\,\frac{2\sin(\alpha\lambda)\cos(\xi\lambda)}{\cosh\lambda}\cos(\lambda\tanh\lambda)^{\frac{1}{2}}\,\tau, \tag{7·2}$$

$$\eta_1 = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\,\frac{2\sin(\alpha\lambda)\cos(\xi\lambda)}{\cosh\lambda}\{(\tanh^2\lambda-\tfrac{1}{2})\cos(\lambda\tanh\lambda)^{\frac{1}{2}}\,\tau + \tfrac{1}{4}\tau(\lambda\tanh\lambda)^{\frac{1}{2}}\operatorname{sech}^2\lambda\sin(\lambda\tanh\lambda)^{\frac{1}{2}}\,\tau\}. \tag{7·3}$$

When ξ and τ are comparable and both large compared with unity, while $\alpha \ll 1$, the integral (7·2) becomes one which has been approximately evaluated by Hendrickson (1962). Takahasi (1947) has also treated a form of (7·2) approximately for $\alpha \gg 1$, that is, when the block width is very large compared with the liquid depth; however, the Hendrickson method would apply equally well to this case, if we were to start from (6·6) instead and work with τ and β. We here describe Hendrickson's method, applied to (7·2), and then apply it to (7·3) also. For simplicity of description of the method, we assume $\alpha \ll 1$; later, analogous results will be derived for rather larger values of α.

By a Dirichlet theorem, if we prescribe a suitable lower limit, say 1, instead of 0 for the integral (7·2), then as the parameters ξ and τ increase without limit, separately or together, the integral tends to zero like $1/\xi$ or $1/\tau$. Physically, this is because interference between neighbouring crests and troughs of the integrand becomes very marked as ξ (or τ) increases. So we may consider the integral from 0 to 1 for the moment. In $0 < \lambda < 1$, we can expand the integrand into a power series in a special way; the two cosine terms, which contain the two large and important parameters, are left as cosines but their argument is put into power series to two terms; the rest of the integrand is cast directly in this form, because the significant contributions to the integral come from a neighbourhood $\lambda = O(1/\xi, 1/\tau)$ of the origin and ξ, τ are both large compared with unity, so that λ is small. Since ξ, τ are also large compared with α, and $\alpha \ll 1$, the factor $\sin(\alpha\lambda)$, which is $O(\alpha/\xi, \alpha/\tau)$ in this neighbourhood of

λ, can be treated as a second-order small quantity approximately equal to $(\alpha\lambda)$, so that we have first

$$\eta_0 \doteqdot \frac{\alpha}{\pi}\int_0^1 [\cos\{\xi\lambda-(\lambda\tanh\lambda)^{\frac{1}{2}}\tau\}+\cos\{\xi\lambda+(\lambda\tanh\lambda)^{\frac{1}{2}}\tau\}]\frac{d\lambda}{\cosh\lambda}. \tag{7·4}$$

Since the argument of the second cosine in (7·4) has no turning point on the positive axis, for $\xi, \tau > 0$, we can either apply Riemann's lemma or integrate by parts to show that this second term is $O(1/\xi)$. (Hendrickson expands in power series and retains only the first term; this is satisfactory for $0 < \lambda < 1$ if $\tau \doteqdot \xi$, but the above seems more general.)

In (7·4) we put

$$\sigma = \tau/\xi = (gh)^{\frac{1}{2}}t/x. \tag{7·5}$$

Since ξ and τ are comparably large, as mentioned earlier, σ will take values in a neighbourhood of 1. We shall soon see that σ must be restricted to be less than about 2 for this method to be valid.

In order to take account of the initial variation of $(\cosh\lambda)^{-1}$, we expand this function, as well as the cosine argument, in a power-series, retaining the first two terms only, and then rearrange the integral in the form of an Airy function:

$$\begin{aligned}\eta_0 &= \frac{\alpha}{\pi}\int_0^1 (1-\tfrac{1}{2}\lambda^2+\dots)\cos\xi\{\lambda(1-\sigma)+\tfrac{1}{6}\sigma\lambda^3+\dots\}\,d\lambda\\ &= \frac{\alpha}{\pi\sigma}\int_0^1 \cos\xi\{\lambda(1-\sigma)+\tfrac{1}{6}\sigma\lambda^2\}\,d\lambda-\frac{\alpha}{\pi\sigma}\int_0^1(1-\sigma+\tfrac{1}{2}\sigma\lambda^2)\cos\xi\{\lambda(1-\sigma)+\tfrac{1}{6}\sigma\lambda^3\}\,d\lambda\\ &= \frac{\alpha}{\pi\sigma}\left(\frac{2}{\xi\sigma}\right)^{\frac{1}{3}}\int_0^\infty \cos\left\{\tfrac{1}{3}u^3+\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{3}}(1-\sigma)\,u\right\}du+O\left(\frac{1}{\xi}\right)-\frac{\alpha}{\pi\sigma\xi}\sin\xi\,(1-\sigma+\tfrac{1}{6}\sigma)\\ &= \frac{\alpha}{\pi\sigma}\left(\frac{2}{\xi\sigma}\right)^{\frac{1}{3}}\mathrm{Ai}\left\{\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{3}}(1-\sigma)\right\}+O\left(\frac{1}{\xi}\right),\end{aligned} \tag{7·6}$$

since (Jeffreys & Jeffreys 1956, §17.07)

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty \cos(\tfrac{1}{3}u^3+uz)\,du. \tag{7·7}$$

Hendrickson (1962) has plotted the function (7·6) graphically, using the separate representations for positive and negative argument of the Airy function in terms of Bessel functions of order $\pm\frac{1}{3}$ (Watson 1922, §6·4). The Airy function Ai (z) is also tabulated for all real values of z by Miller (1946). In order to see physically what happens as $\sigma = \tau/\xi$ varies from small to fairly large values (compared with 1), we content ourselves here with noting the asymptotic expansions for large positive or large negative values of the argument (Watson 1922, §7.23)

$$\mathrm{Ai}(z) \sim \frac{1}{2\pi^{\frac{1}{2}}z^{\frac{1}{4}}}\exp(-\tfrac{2}{3}z^{\frac{3}{2}}), \tag{7·8}$$

$$\mathrm{Ai}(-z) \sim \frac{1}{\pi^{\frac{1}{2}}z^{\frac{1}{4}}}\sin(\tfrac{2}{3}z^{\frac{3}{2}}+\tfrac{1}{4}\pi) \quad (z \gg 1). \tag{7·9}$$

Now, for $\sigma < 1$, that is, from (7·5)

$$x > (gh)^{\frac{1}{2}}t \tag{7·10}$$

the argument $(2\xi^2/\sigma)^{\frac{1}{3}}(1-\sigma)$ is positive and tends to infinity as $\sigma \to 0$. Hence, by (7·8), the disturbance is exponentially small. In fact Ai (z) is monotonic decreasing for $z > 0$, so that the disturbance begins effectively at $x = (gh)^{\frac{1}{2}} t$, that is, the head of the wave propagates with the velocity $(gh)^{\frac{1}{2}}$ of waves long compared with the depth.

After the first swing, the Airy function decays in amplitude and behaves like (7·9) for z less than about -2. When $\sigma > 1$, the argument is negative and tends to $-\infty$ as σ increases; then (7·6) and (7·10) give

$$\eta_0 \sim \frac{2^{\frac{1}{4}}\alpha}{\pi^{\frac{1}{2}}\sigma^{\frac{3}{4}}\xi^{\frac{1}{2}}} \sin\left\{\frac{2}{3}\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{2}}(\sigma-1)^{\frac{3}{2}}+\tfrac{1}{4}\pi\right\}. \tag{7·11}$$

However, the growth of σ is restricted by the necessity for the point of stationary phase of (7·6) to be given by a value of λ within the approximate range of integration, $0 < \lambda < 1$, in order that the power-series approximation shall hold good; the stationary point is

$$\lambda = \{2(\sigma-1)/\sigma\}^{\frac{1}{2}}$$

and hence σ should be roughly between 1 and 2. For $\sigma < 1$, the stationary point recedes to the negative λ axis, there is no dominant term and (7·8) is appropriate, as remarked above; but for $\sigma > 2$ the stationary point lies in $\lambda > 1$ and the extension of the range to $0 < \lambda < \infty$ in (7·6) will include it and introduce spurious effects. This restriction was not noted by Hendrickson.

We shall expect the method to break down when $\sigma \gg 1$ since then the behaviour of the cosine argument in (7·4) is not well represented by the power-series development to two terms. This inequality means that a time long compared with the travel time of the leading wave has elapsed, and we are well into the tail or 'coda' of the disturbance. But when $\tau \gg \xi$, the integrals (7·2) and (7·3) can be attacked by a more powerful method (this is done in the next section), and we find that the numerical values of the constants are rather different, while the amplitude decays like $\tau^{-\frac{1}{2}}\exp(-y_1\tau)$, where y_1 is a positive constant, which feature is not exhibited by (7·11). We therefore conclude that the results of this Section are valid only when σ is not too large.

We observe that near the leading wave $\sigma = 1$, η_0 is $O(\xi^{-\frac{1}{3}})$, and when σ is a little larger, say, for the next nine or ten waves, by (7·11) η_0 is $O(\xi^{-\frac{1}{2}})$. Thus the term retained in (7·4) and (7·6) is indeed larger than any of the terms neglected, which are $O(\xi^{-1})$.

One of the restrictions placed by Hendrickson on the data can be removed, namely the condition that α is to be small compared with unity. If we demand only that α be small compared with ξ and τ, so that within the significant neighbourhood $\lambda = O(1/\xi, 1/\tau)$ of the origin the factor $\sin(\alpha\lambda)$ is still $O(\alpha/\xi, \alpha/\tau)$ but not necessarily of the second-order of small quantities, we can take account of $\sin(\alpha\lambda)$ as well as $\cosh\lambda$, and we find

$$\eta_0 = \frac{\alpha}{\sigma}\{1+(1-\sigma)\tfrac{1}{3}\alpha^2\}\left(\frac{2}{\xi\sigma}\right)^{\frac{1}{3}} \mathrm{Ai}\left\{\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{3}}(1-\sigma)\right\}. \tag{7·12}$$

Thus, considering the effect of the block to this order for $\alpha = O(1)$ makes little difference near the leading wave $\sigma = 1$, but will cause the amplitude to fall off as σ increases beyond the value 1. This may be taken as the effect of destructive interference between the two wave trains which are generated from the front and back edge of the block; if we start from (6·9), giving $\bar{\eta}_0$, we obtain an analogue of (7·6) involving the integral of the Airy function, and the two such expressions arising from $\beta = \alpha \pm \xi$ represent two superposed wave systems which interfere.

The formula (7·6) is in fact a good approximation for $\alpha \leqslant \frac{1}{2}$ as stated by Hendrickson, since then in (7·12), $\frac{1}{3}\alpha^2 < \frac{1}{10}$.

We now proceed to examine the effect of compressibility, exemplified by (7·3), near the leading gravity wave. For $\alpha \ll 1$, the same kind of rearrangement as in the working of (7·6) leads to

$$\eta_1 = -\frac{\alpha}{2}\frac{5-4\sigma}{\sigma}\left(\frac{2}{\xi\sigma}\right)^{\frac{1}{3}} \mathrm{Ai}\left\{\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{3}}(1-\sigma)\right\} + \frac{\alpha\tau}{4}\frac{10-7\sigma}{3\sigma}\left(\frac{2}{\xi\sigma}\right)^{\frac{2}{3}} \mathrm{Ai}'\left\{\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{3}}(1-\sigma)\right\}, \quad (7·13)$$

on use of the formal result

$$\mathrm{Ai}'(z) = -\frac{1}{\pi}\int_0^\infty u \sin\left(\tfrac{1}{3}u^3 + uz\right) \mathrm{d}u. \quad (7·14)$$

Now, comparison with η_0 as in (7·6) leads to an anomaly at $\sigma = 1$. For any fixed σ, the first term in η_1 bears the ratio (gh/c^2) to η_0; but when $\sigma = 1$ (taking this value to keep the Airy function arguments fixed at zero), as ξ increases the second term increases relatively to η_0; putting $\tau = \xi$, the ratio of $(\eta_1\, gh/c^2)$ to η_0 is

$$\left|\frac{2^{\frac{1}{3}}\mathrm{Ai}'(0)}{4\,\mathrm{Ai}(0)}\right| \frac{gh}{c^2}\xi^{\frac{2}{3}} = (0{\cdot}230)\frac{gh}{c^2}\xi^{\frac{2}{3}}. \quad (7·15)$$

Consequently, as ξ increases, this ratio increases as $\xi^{\frac{2}{3}}$ and will eventually exceed unity. But we know that gravity, and not compressibility, is the dominating influence at the head of this disturbance. Thus we are forced to conclude that $\sigma = 1$ no longer represents the head of the disturbance exactly, that is, the velocity of the leading wave (and the associated value of σ) is slightly affected by compressibility so that, when ξ is sufficiently large, the scale of the wave is such that the point $\sigma = 1$ is at a significant distance from the leading wave.

To determine what the effect on the velocity is, to the first order in (gh/c^2), we try to combine (7·6) and (7·13) into a single formula which gives these expressions for η_0 and η_1 as coefficients when it is expanded in powers of (gh/c^2), being thus a kind of generating function for η_0, η_1, and which more clearly depicts the effect of this parameter on the motion. The form of the second term in (7·13), with Ai′, suggests a representation as the first two terms of a Taylor series (we might call it Taylor synthesis). Thus inserting (7·6) and(7·13), (7·1) becomes to order (gh/c^2)

$$\begin{aligned}\eta\, \mathrm{e}^{-\gamma h} &= \eta_0 + \frac{gh}{c^2}\eta_1 \\ &= \alpha\left(\frac{2}{\xi\sigma}\right)^{\frac{1}{3}}\frac{1}{\sigma}\left(1-\frac{5-4\sigma}{2}\frac{gh}{c^2}\right)\left[\mathrm{Ai}\left\{\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{3}}(1-\sigma)\right\}\right. \\ &\qquad\qquad \left.+\frac{gh}{c^2}\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{3}}\frac{\sigma(10-7\sigma)}{12}\mathrm{Ai}'\left\{\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{3}}(1-\sigma)\right\}\right] \\ &= \alpha\left(\frac{2}{\xi\sigma}\right)^{\frac{1}{3}}\frac{1}{\sigma}\left(1-\frac{gh}{c^2}\frac{5-4\sigma}{2}\right)\mathrm{Ai}\left[\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{3}}\left\{1-\sigma+\frac{gh}{c^2}\frac{\sigma(10-7\sigma)}{12}\right\}\right].\end{aligned} \quad (7·16)$$

Equation (7·16) is now the required single formula. The head of the disturbance still corresponds to the value of σ which causes the argument of the Airy function to vanish:

$$1-\sigma+\frac{gh}{c^2}\frac{\sigma(10-7\sigma)}{12} = 0. \quad (7·17)$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

A first approximation is $\sigma = 1$. Substituting this into the small term, the approximation to order (gh/c^2) is

$$\sigma = 1 + \tfrac{1}{4}gh/c^2. \tag{7·18}$$

So, from (7·5), the head of the disturbance is associated with a value of x/t (velocity) with

$$x/t = (gh)^{\frac{1}{2}}(1 - \tfrac{1}{4}gh/c^2). \tag{7·19}$$

Thus, the effect of compressibility on the head of the long gravity wave at large ξ is to decrease the leading wave velocity by a factor $(1-\frac{1}{4}gh/c^2)$; by (7·16) the amplitude of the leading wave is decreased by a factor $(1-\frac{1}{2}gh/c^2)$, and if gh/c^2 is small this exactly counteracts the effect of the factor $e^{\gamma h}$ so that amplitude variation at the head is at most of the second order in gh/c^2.

We see that in this formulation, the troublesome point $\sigma = 1$ gives the Airy function in (7·16) the value

$$\mathrm{Ai}\{(2\xi^2)^{\frac{1}{3}}\tfrac{1}{4}gh/c^2\}, \tag{7·20}$$

so that, as ξ increases, the disturbance is exponentially small by (7·8). This point $\sigma = 1$ is not in the region of the gravity disturbance proper—from (7·18), this region is

$$\sigma > 1 + \tfrac{1}{4}gh/c^2.$$

Stoneley (1963) has given a brief treatment of the compressibility approximation for the problem of an initial *surface* elevation of block type, for example, a block dropped into water with bottom face horizontal. He makes an approximate modification of the incompressible-fluid solution given by Jeffreys & Jeffreys (1956, §17.09), and derives a corresponding Airy function and hence a correction to the speed of the leading wave

$$x/t = (gh)^{\frac{1}{2}}(1 - gh/c^2)^{\frac{1}{2}} \doteqdot (gh)^{\frac{1}{2}}(1 - \tfrac{1}{2}gh/c^2) \tag{7·21}$$

in our notation. Thus in Stoneley's problem the increment in wave slowness is twice the increment in the present problem. This is introduced by the different forms of integrals representing the solution; for example, the factor $1/\cosh\lambda$ does not appear in Stoneley's integrand, and this leads in our problem to an extra factor $1/\sigma$ to modify the wave attenuation as σ increases. We obtain different results for surface disturbances and bed disturbances; a not unexpected phenomenon.

The formula (7·16) is derived for $\alpha \ll 1$; but by the same method as was used to obtain (7·12), we can find the effect of the finite block width for rather larger values of α. The working is the same, and we find that the analogous equation to (7·12) is

$$\eta = e^{\gamma h}\alpha\left[\frac{1}{\sigma}\{1+(1-\sigma)\tfrac{1}{3}\alpha^2\} - \frac{1}{2}\frac{gh}{c^2}\left\{1+\frac{1-\sigma}{\sigma}(5+\tfrac{1}{3}\alpha^2)\right\}\right]\left(\frac{2}{\xi\sigma}\right)^{\frac{1}{3}}$$
$$\times \mathrm{Ai}\left\{\left(\frac{2\xi^2}{\sigma}\right)^{\frac{1}{3}}\left[1-\sigma+\frac{1}{4}\frac{gh}{c^2}\sigma^2\frac{1+(10+\alpha^2)(1-\sigma)/3\sigma}{1+(1-\sigma)\alpha^2/3}\right]\right\}. \tag{7·22}$$

If α^2 is neglected we obtain (7·16), and if (gh/c^2) is neglected (7·12) appears.

Since the terms in α^2 also contain a factor $(1-\sigma)$, to this order in (gh/c^2) the block width has no effect on the velocity of the leading wave, which is still given by (7·19).

8. The coda of the gravity wave

In this section we analyse the residual gravity disturbance which prevails at times large compared with the arrival time of the leading wave discussed in the last section. At such times, $\sigma \gg 1$, $\tau \gg \xi$, we expect that the true behaviour of the expression $(\lambda \tanh \lambda)^{\frac{1}{2}}$ in (7·2) and (7·4) will become important in the evaluation of the integrals; the error in replacing it by its power series and then cutting it off at the second term increases with σ.

We can attack this problem for unrestricted values of ξ as far as lower bounds are concerned; the only condition we now impose is that τ shall be large compared with α, ξ and unity. This ensures that the leading waves from both edges have already passed, and also that the variation in $\cos(\lambda \tanh \lambda)^{\frac{1}{2}}\tau$ will outweigh the variation of terms like $1/\lambda$ and $\cosh \lambda$, as well as of $\sin(\beta\lambda)$.

With these facts in mind, we may return to (6·6) which gives η_X as a function of β. We begin with a consideration of (6·9):

$$\bar{\eta}_0 = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\lambda}\frac{\sin(\beta\lambda)}{\cosh \lambda}\cos(\lambda \tanh \lambda)^{\frac{1}{2}}\tau = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{d\lambda}{\lambda}\frac{\sin(\beta\lambda)}{\cosh \lambda}\exp\{i(\lambda \tanh \lambda)^{\frac{1}{2}}\tau\}. \tag{8·1}$$

We now apply the saddlepoint technique to (8·1). The saddle points for an integral with $\exp\{f(\lambda)\,\tau\}$ are the zeros of $f'(\lambda)$; here

$$f'(\lambda) = i\,\frac{\lambda + \frac{1}{2}\sinh 2\lambda}{2\cosh^2 \lambda\,(\lambda \tanh \lambda)^{\frac{1}{2}}}. \tag{8·2}$$

We remark that if we had included $\sin(\beta\lambda)$ in the exponential in (8·1), we would have had an extra term $\pm i\beta/\tau$ in the expression (8·2) for $f'(\lambda)$. For general values of β/τ this would require extensive numerical computations to find the saddle points as functions of this parameter; and as we have the powerful method of § 7 to obtain asymptotic expansions for β/τ around the value 1, this work would not tell us much. But then $\tau \gg \beta$, the Hendrickson approximation for $(\lambda \tanh \lambda)^{\frac{1}{2}}\tau$ will be in error by an amount significant compared with $(\beta\lambda)$, as explained above, and the method becomes dubious; it is just in this instance that we can neglect β/τ and conveniently simplify the problem by using equation (8·2) as it stands. The saddle points in the complex λ plane are now given by

$$2\lambda + \sinh 2\lambda = 0 \quad (\lambda \neq 0). \tag{8·3}$$

The equation (8·3) has no roots on the real or imaginary axis. By separating (8·3) into real and imaginary parts, we find that the roots are symmetrically placed in sets of four with regard to the axes, that when $\mathscr{I}\lambda > 0$ the solutions lie in regions $(N+\frac{1}{2})\pi < \mathscr{I}\lambda < (N+\frac{3}{4})\pi$ where N is a positive integer or zero, and that the first pair of saddle points (corresponding to $N = 0$) are at

$$\lambda_\pm = \pm(1\cdot 1251) + i(2\cdot 1062). \tag{8·4}$$

The computation of (8·4) was performed manually by a method of successive approximation, using a set of tables. The saddle points are taken in the upper half λ plane since $(\lambda \tanh \lambda)^{\frac{1}{2}}$ has positive imaginary part in this half-plane. The lowest set of points $(N = 0)$ is taken to avoid crossing the isolated essential singularities at the zeros $\lambda = (N+\frac{1}{2})\pi i$ of $\cosh \lambda$. The Kelvin contour of stationary phase

$$\mathscr{I}(\lambda \tanh \lambda)^{\frac{1}{2}} = \mathscr{I}(\lambda_\pm \tanh \lambda_\pm)^{\frac{1}{2}} = 0\cdot 6863 \tag{8·5}$$

is found to be suitable for the problem; it crosses the imaginary axis at i(0·744), which is between the real axis and the first isolated essential singularity, $\lambda = \frac{1}{2}i\pi$, so that no contribution arises therefrom, and by Jordan's lemma and the choice of half plane there is no contribution from the arcs at infinity connecting the contour to the real axis. The contour of stationary phase is shown in figure 11.

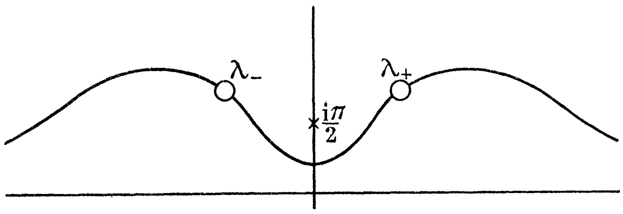

FIGURE 11. The contour of stationary phase in the λ plane.

For positive β larger than about unity, the combined stationary phase contributions to (8·1) are approximately

$$\bar{\eta}_0 = \frac{0{\cdot}1455}{\tau^{\frac{1}{3}}} e^{-y_1\tau + x_1\beta} \cos(y_2\tau - x_2\beta - z_1), \tag{8·6}$$

where x_1, x_2, y_1, y_2 are positive numbers associated with the saddle points:

$$\left.\begin{aligned} x_1 &= 2{\cdot}1062, \quad x_2 = 1{\cdot}1251, \quad \lambda_+ = x_2 + ix_1 \\ y_1 &= 0{\cdot}6863, \quad y_2 = 1{\cdot}476 \end{aligned}\right\} \tag{8·7}$$

and $z_1 = 0{\cdot}694$.

For incompressible fluid, the coda of the gravity wave is seen from (8·6) to be an exponentially damped progressive wave with velocity

$$(y_2/x_2)(gh)^{\frac{1}{2}} = (1{\cdot}31)(gh)^{\frac{1}{2}} \tag{8·8}$$

and wavelength
$$(2\pi/x_2)\,h \doteqdot (5{\cdot}6)\,h \tag{8·9}$$

or 5·6 times the depth.

There are two such waves, one corresponding to each edge of the block. For a point over the block, we put $\beta = \alpha+\xi$, $\alpha-\xi$, and add; in the half $0<\xi<\alpha$ the contribution from $\beta = \alpha+\xi$ will be more important because of the exponential factor. For points not over the block, $\xi > \alpha$, we subtract the result for $(\xi-\alpha)$ from that for $(\xi+\alpha)$; if ξ is not far removed from α, the contribution from $\beta = \xi+\alpha$ is again dominant. But if $\alpha \ll \xi$, still with $\xi \ll \tau$, we can write

$$\begin{aligned} &e^{x_1(\xi+\alpha)} \cos\{y_2\tau - x_2(\xi+\alpha) - z_1\} - e^{x_1(\xi-\alpha)} \cos\{y_2\tau - x_2(\xi-\alpha) - z_1\} \\ &\quad \doteqdot 2\alpha \frac{\partial}{\partial\xi}\{e^{x_1\xi} \cos(y_2\tau - x_2\xi - z_1)\} \\ &\quad = 2\alpha(x_1^2+x_2^2)^{\frac{1}{2}} e^{x_1\xi} \sin(y_2\tau - x_2\xi - z_1 + \tan^{-1} x_1/x_2). \end{aligned} \tag{8·10}$$

Then
$$\eta_0 = (0{\cdot}695)\frac{\alpha}{\tau^{\frac{1}{3}}} e^{-y_1\tau + x_1\xi} \sin(y_2\tau - x_2\xi + 0{\cdot}386). \tag{8·11}$$

This result could also have been obtained by putting $\sin(\alpha\lambda) \doteqdot \alpha\lambda$ in the integral for η_0, as in §7. The essential characteristics (wavelength and velocity) of the disturbance are again given by (8·8) and (8·9).

We finally turn to the effect of compressibility on the coda. This is again given by (6·10) here; for $\tau \gg 1$, the dominant term here is the second (containing the factor τ), and this integral may be likewise evaluated at the saddle points $\lambda_\pm$ to give

$$\bar{\eta}_1 = \frac{\tau}{4}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda}\frac{\sin(\beta\lambda)}{\cosh^3\lambda}(\lambda\tanh\lambda)^{\frac{1}{2}}\exp\{i(\lambda\tanh\lambda)^{\frac{1}{2}}\tau\}$$
$$= (0·0275)\,\tau^{\frac{1}{3}}\exp\{-y_1\tau+x_1\beta\}\cos(y_2\tau-x_2\beta+z_2), \quad (8·12)$$

with $$z_2 = 0·078. \quad (8·13)$$

We now observe the same kind of anomalous behaviour of the ratio of $\bar{\eta}_1 gh/c^2$ to $\bar{\eta}_0$ as we found between the corresponding terms in the preceding section, discussed under (7·13); the ratio is here $O(\tau gh/c^2)$, and increases without limit as $\tau \to \infty$. But we can overcome this difficulty by use of the same technique; we combine $\bar{\eta}_0$ and $\bar{\eta}_1$ into a single formula by regarding them as the leading terms in a Taylor expansion. We have, from (6·7)

$$\eta_X = \bar{\eta}_0+\frac{gh}{c^2}\bar{\eta}_1 = \frac{0·1455}{\tau^{\frac{1}{3}}}\exp\{-y_1\tau+x_1\beta\}\left\{\cos(y_2\tau-x_2\beta-z_1)+Z\tau\frac{gh}{c^2}\cos(y_2\tau-x_2\beta-z_1+z_3)\right\} \quad (8·14)$$

with $$z_3 = 0·7721, \quad Z = 0·1897. \quad (8·15)$$

If this is the Taylor expansion of a first-order correction formula

$$\eta_x = \frac{0·1455}{\tau^{\frac{1}{3}}}\exp\left\{-y_1\tau\left(1+\frac{gh}{c^2}\delta_1\right)+x_1\beta\right\}\cos\left\{y_2\tau\left(1+\frac{gh}{c^2}\delta_2\right)-x_2\beta-z_1\right\}$$
$$= \frac{0·1455}{\tau^{\frac{1}{3}}}\exp\{-y_1\tau+x_1\beta\}\Big[\cos(y_2\tau-x_2\beta-z_1)$$
$$-\tau\frac{gh}{c^2}\{y_1\delta_1\cos(y_2\tau-x_2\beta-z_1)+y_2\delta_2\sin(y_2\tau-x_2\beta-z_1)\}\Big], \quad (8·16)$$

then equating cos/sin $(y_2\tau-x_2\beta-z_1)$ in (8·14) and (8·16),

$$\left.\begin{aligned}\delta_1 &= -Z\cos z_3/y_1 = -0·20,\\ \delta_2 &= Z\sin z_3/y_2 = 0·09.\end{aligned}\right\} \quad (8·17)$$

We may substitute all the numerical values from (8·7), (8·15) and (8·17) in (8·16), and then

$$\eta_X = \frac{0·1455}{\tau^{\frac{1}{3}}}\exp\left[-(0·6863)\,\tau\left\{1-(0·20)\frac{gh}{c^2}\right\}+(2·0162)\,\beta\right]$$
$$\times\cos\left[(1·476)\,\tau\left\{1+(0·09)\frac{gh}{c^2}\right\}-(1·1251)\,\beta-(0·694)\right]. \quad (8·18)$$

The velocity of the progressive wave is now

$$\frac{y_2}{x_2}(gh)^{\frac{1}{2}}\left(1+\frac{gh}{c^2}\delta_2\right) = (1·31)\,(gh)^{\frac{1}{2}}\left\{1+(0·09)\frac{gh}{c^2}\right\}. \quad (8·19)$$

The effect of compressibility on the coda is that the velocity is increased by about (0·12) $(gh)^{\frac{1}{2}}(gh/c^2)$ and that the decay coefficient in τ is reduced by about (0·14) (gh/c^2).

By the same process as that used for deriving (8·11), if $\alpha \ll \xi \ll \tau$,

$$\eta = (0·695)\frac{\alpha}{\tau^{\frac{1}{3}}}\exp\left\{-y_1\tau\left(1+\frac{gh}{c^2}\delta_1\right)+x_1\xi\right\}\sin\left\{y_2\tau\left(1+\frac{gh}{c^2}\delta_2\right)-x_2\xi+(0·386)\right\}. \quad (8·20)$$

We see that the coda exhibits different properties from the tail of the Airy function discussed in § 7; the coda is a non-dispersive progressive wave and is exponentially damped in time, whereas the asymptotic expansion of (7·6) for large $\sigma = \tau/\xi$ gave an inverse power law only. There is thus a transition from a profile (7·16) to a profile (8·20) as σ increases from 0 through 1 to infinity.

The increase in velocity of the coda due to compressibility is remarkable in comparison with the decrease in velocity of the head wave.

Conclusion

The formal solution of the problem set up in this paper is the double integral (1·21), but—as is quite usual in wave propagation problems—this is only the beginning of the road towards physical interpretation of the solution. This task is lightened by the separation of the disturbance into two parts, acoustic and gravity-controlled; in each type, the other effect assumes the role of a small perturbation, and is discussed as such together with the principal phenomenon. In both perturbation studies, the parameter (gh/c^2) is of paramount importance.

As mentioned in the Introduction and developed in §§ 2 to 4, the acoustic waves, travelling with the sound velocity c, are of two kinds; one kind is derived from each of the two edges of the block, and is continuous, suffering only steep rises and falls at the arrival times (these are derived from geometrical ray theory) and exists everywhere, while the other kind is a series of direct arrivals observed only over the block, resembling the effect of a piston in a pipe, and suffers jumps which are discontinuous in time and are twice the amplitude of the initial disturbance (because there is an incident and reflected wave). The arrival times are governed by the (still simpler) geometry of the problem. The signs in all three sets alternate because of changes of phase at the bottom.

Gravity terms, small in the first stages, assume more and more importance as time goes on (at a fixed point), and finally dominate in the manner observed in most ocean wave studies, including tsunamis, which can be generated by such means as described here. For slightly compressible fluid, the initial gravity wave is determined and numerically calculated in § 6, and the compression perturbation is inserted; this result is expected to be valid for times large compared with the travel time of the first acoustic pulse, but small compared with the travel time of long waves (for which $kh \ll 1$, typical of tsunamis)—or, certainly, not large. It is thus restricted in application, but is still of interest. In a bridge passage (§ 5) we show that the sum of all the acoustic waves, in the limit of zero time and infinite sound velocity, reduces to the initial surface elevation for the problem with incompressible fluid, so that this initial elevation is to be regarded as primarily an acoustic effect.

At large distances (§ 7) the leading gravity wave propagates with the long-wave velocity $(gh)^{\frac{1}{2}}$ $(\ll c)$, and behaves like an Airy function which is small in front and sinusoidal behind (compare the first arrival of a tsunami); as time proceeds, this wave becomes a non-dispersive progressive wave with an exponential tail (the coda) which is not the Airy function limit, so that there is yet another transitional stage (§ 8). The correction to first order for compressibility in the velocity of the head wave is of the same sign as, but smaller than, that found by Stoneley (1963) for a different tsunami generation problem (using a different method). It is a decrease in the velocity. On the other hand, the correction in the coda represents an increase in velocity. This suggests that some of the energy in the disturbance is trapped in the acoustic reflexion zone and in some way transferred from the leading wave of incompressible fluid, and spread out over the coda.

This paper is abridged from the author's doctoral thesis. During its preparation the author was supported by a grant from the Department of Scientific and Industrial Research.

References

Bondi, H. 1947 Waves on the surface of a compressible liquid. *Proc. Camb. Phil. Soc.* **43**, 73–95.

Cagniard, L. 1939 *Reflections et Refractions des Ondes Séismiques Progressives* (Thèse). Paris: Gauthier–Villars et Cie (English Translation: McGraw Hill, 1962).

Friedlander, F. G. 1958 *Sound Pulses*, §6·4. Cambridge University Press.

Hendrickson, J. A. 1962 *See* Wilson, B. W. *et al.* (1962).

Jeffreys, H. 1931 The formation of Love waves in a two-layer crust. *Gerl. Beitr. Geophys.* **30**, 336–350.

Jeffreys, H. & Jeffreys, B. S. 1956 *Methods of mathematical physics*. Cambridge University Press.

Jeffreys, H. & Sells, C. C. L. 1963 SH from a source in the lower layer. *Geophys. J.* **7**, 593–603.

Lamb, H. 1957 *Hydrodynamics* (6th ed.), §311. Cambridge University Press.

Lapwood, E. R. 1949 The disturbance due to a line pulse in a semi-infinite medium. *Phil. Trans.* A, **242**, 63–100.

Miller, J. C. P. 1946 *British Association Mathematical Tables*, part-volume B, *The Airy integral*. Cambridge University Press.

Munk, W. H. 1962 Long ocean waves (6. Tsunamis), from *The sea* (ed. M. N. Hill), pp. 658–63. Interscience.

Pidduck, F. B. 1912 The wave problem of Cauchy and Poisson for finite depth and slightly compressible fluid. *Proc. Roy. Soc.* A, **86**, 396–405.

Stoneley, R. 1926 The effect of the ocean on Rayleigh waves. *Mon. Not. R. Astr. Soc.* (*Geophys. Suppl.*), **1**, 349–356.

Stoneley, R. 1963 The propagation of tsunamis. *Geophys. J.* **8**, 64–81.

Takahasi, R. 1947 On seismic sea waves caused by deformations of the sea bottom. 3. The one-dimensional source. *Bull. Earth. Res. Inst.* (*Tokyo*), **25**, 5–8.

Unoki, S. & Nakano, M. 1953 On the Cauchy–Poisson waves caused by the eruption of a submarine volcano. *Pap. Met. Geophys.* **4**, 139–150.

Watson, G. N. 1922 *Theory of Bessel functions*. Cambridge University Press.

Wilson, B. W. Webb, L. M. & Hendrickson, J. A. 1962 The nature of tsunamis; their generation and dispersion in water of finite depth. *U.N.E.S.C.O. Tech. Rep.* no. SN 57–2. Webb, L. M.: Appendix 1. Theory of waves generated by surface and sea bed disturbances. Hendrickson, J. A.: Appendix 2. Asymptotic solution to surface profile of tsunamis created by rectangular upthrust of sea bed at large distances.